Consecutive Numbers

Madhav Kaushish

November 23, 2017

Learning Outcomes:

1. Coming up with conjectures
2. Coming up with proofs
3. Generalising theorems

The following is a dialogue between a teacher and a student:

T: Give me two consecutive number.

S: 4 and 5

T: Add them. What do you get?

S: 9

T: Give me a few more consecutive number pairs

S: 9 and 10, 18 and 19, and 43 and 44

T: Add each of them

S: You get 19, 37 and 87 respectively

T: Do you see a pattern?

S: Not sure

T: Try a few more then

S: 12 and 13, 145 and 146, 33 and 34

T: Add each of them

S: 25, 191 and 67

T: Let's write all of them together:

Consecutive Numbers	Sum
4 and 5	9
9 and 10	19
18 and 19	37
43 and 44	87
12 and 13	25
145 and 146	291
33 and 34	67

T: Now do you see a pattern?

S: Yes! All the sums are odd numbers.

T: That is true here, but is it true for all pairs of consecutive numbers?

S: Seems like it is. But we can't be sure. We can't try out every number!

T: What you have is a conjecture. A conjecture is a statement which is either true or false, but we don't know which one it is currently. If you prove it true it becomes a theorem.

S: How do we prove it?

T: What you need here is an argument which will work for all numbers

S: How do we go about creating such an argument?

T: In this case, one way we can go about it is to notice that if we add 1 to an odd number, you get an even number, and if we add 1 to an even number, we get an odd number.

S: So, if we can show that 1 less than the sum of two consecutive numbers is even, we automatically get that the sum of two consecutive numbers is odd.

T: Right!

S: How do we do this now?

T: What is one less than the sum of two consecutive numbers?

S: Hmm... that would be the sum of the smaller number with itself, wouldn't it?

T: Yes!

S: How does that help us?

T: You want to show that the sum of a number with itself is even, right?

S: Right

T: So, what is an even number?

S: Now I get it! An even number is a number which can be broken into two equal parts. Obviously, if we add a number to itself, we will be able to break the result into two equal parts. So, the result is even! And, one more than it is odd, so the sum of two consecutive numbers is odd.

T: Good job! You now have your first theorem. But this is only the start. Don't rest on your laurels

S: What do we do next?

T: Now, that you have proved something, you shouldn't stop here. You should look at the Theorem you have created and try asking questions related to it.

S: Such as?

T: Now, that you answered a question about two consecutive numbers, how about trying to add three consecutive numbers?

S: Let's try that out.

S: Some seem to be odd and some seem to be even, so that's not a pattern here.

T: What about other patterns?

S: Are all of the results divisible by 3?

T: I think all the ones you have tried out so far are. But do check it yourself

S: Yes, they are all divisible by 3. But I'm not sure if that pattern will continue

Consecutive Numbers	Sum
3, 4 and 5	12
8, 9 and 10	27
17, 18 and 19	54
42, 43 and 44	129
11, 12 and 13	36
144, 145 and 146	435
32, 33 and 34	99

T: Try out a few more examples in order to figure out whether your conjecture, that the pattern will continue, is plausible or not.

S: What do you mean by plausible?

T: A plausible conjecture is one which you are convinced is true but have not proved so far

S: After trying some more examples, we are satisfied that this conjecture is plausible

T: Now you need a proof!

S: This time, let us make the argument. Ok, so, let's try a similar method to the last one. Adding three consecutive numbers is like adding the smaller number to itself three times and then adding 1, for the second number, and two for the third number.

T: This seems like a promising approach. Put it down as an equation

S: Ok, Sum of three consecutive numbers = 3 x smallest number + 1 + 2. Oh! that is the same as saying Sum of three consecutive numbers = 3 x smallest number + 3. By the distributive law of multiplication over addition, we can say that the Sum of three consecutive numbers = 3 x (smallest number + 1). Since the right hand side of the equation is divisible by 3, the left hand side must also be! So, the sum of three consecutive numbers is divisible by 3!

T: You have your second theorem now!

S: How do we move forward? Should we try four, five and six?

T: Yes you should. However, first, reflect on what you have done so far. Your first theorem showed that the sum of two consecutive numbers is odd. In other words, the sum of 2 consecutive numbers is not divisible by 2. The second theorem showed that the sum of 3 consecutive numbers is divisible by 3. So, as you go forward, you should ask the question, 'Is the sum of 4 consecutive numbers divisible by 4?', 'Is the sum of 5 consecutive numbers divisible by 5?' and so on. Now, that you know what you are looking for, you no longer need to spend time finding patterns like you did earlier.

S: Good point. Now, let's try 4, 5 and 6!

T: Try the same method

—After some work—

S: Using the same methods, we found that the sum of 4 consecutive numbers is not divisibly by 4. The sum of 5 consecutive numbers is divisible by 5 and the sum of 6 consecutive numbers is not divisible by 6.

T: Nice! Now, that you have a bunch of theorems, can you see a pattern amongst those theorems? Now, you are looking for patterns in theorems rather than patterns in numbers!

S: How do we look for those?

T: Write out what you have in the form of a table.

Number of Consecutive Numbers	**Divisible by the Number of Consecutive Numbers?**
2	**No**
3	**Yes**
4	**No**
5	**Yes**
6	**No**

T: Do you see a pattern?

S: It seems to alternate. For even numbers, it is not divisible, but for odd numbers it is.

T: So, let me state what you are saying. You are saying that the sum of n consecutive numbers is divisible by n if n is even, but not divisible by n if n is odd.

S: What is n?

T: n is a variable. A variable is like a place holder. Here, by placing n, you are just saying, 'The sum of any number of consecutive numbers is divisible by the number of consecutive numbers if there are an odd number of consecutive numbers, and is not divisible by the number of consecutive numbers if there are an even number of consecutive numbers.'

S: That is what we are saying. Though using n does make it a lot shorter!

T: Can you attempt to prove this?

S: Let's try using the same methodology. The sum of n consecutive numbers = smallest number x n + 1 + 2 + 3 + ... + (n-1). The first part is always divisible by n. The second part seems really complicated!

T: Separate it out and write only the second part. Since the first part is always divisible by n, now all you need to do is to show that the second part is divisible by n if n is odd, and not divisible by n if n is even.

S: Here's the second part: 1+2+3+4+...+(n-1)

T: Let's leave this here for now. Moving forward from here is quite hard. Spend some time thinking about it. However, if you are unable to get it, google Gauss. He was a German mathematician who came up with a way to add sequences like the one you have. He did this while he was in school!