

Discrete Geometry

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Most of the experience you have probably had with geometry is Euclidean Geometry, where between any two points, there are an infinite number of points. Let us consider an alternative – between any two points, there are a finite number of points.

In this world, we need the concept of neighboring points. For now, we can use our intuition of what that means rather than formally set up the world.

A line in this world would be a collection of points where at most two points have exactly one neighbor and the rest of the points have exactly two neighbors.

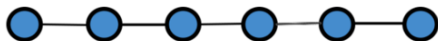
Straight Lines

World 1:

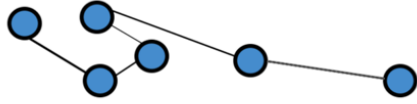
Consider the world where every point apart from two of them have exactly two neighbors, and those two points have exactly one neighbor. This world looks like:



In the above picture, whether two points are neighbors is determined by how close they are to each other. However, that can lead to confusion in more complicated situations, so let us signify two points as neighbors by putting a segment between them:



Now, the above representation is equivalent to:



For convenience, let us call those segments ‘edges’.

In this world, we can ask questions about shapes. For that purpose, let us borrow definitions of shapes from Euclidean Geometry. Let us start with lines and straight lines (in many textbooks of Euclidean Geometry, what is being called a line here is called a curve, and what is being called a straight line is a line).

Definition 1: A line between two points A and B is a connected path between A and B (we will not go into a definition of connected here – for now, you can use your intuition)

Definition 2: A straight line between two points A and B is a shortest line between A and B.

Notice that for straight lines, we need a notion of length of a path/line. In the world we are operating in, an obvious candidate for length of a path is the number of edges you cross when travelling the path. So the path involving two points which are neighbors has length 1.

We need a way to represent lines. Let us first label the points. Then, the line from A to D can be represented by ABCD. Notice there is only one such line, making it a straight line.

So, let us state this as a conjecture: Conjecture 1: All lines in this world are straight lines

Why is this the case? It is because there is only one line starting at a point X and ending at a point Y. Since a straight line is a shortest line, this line must be straight.

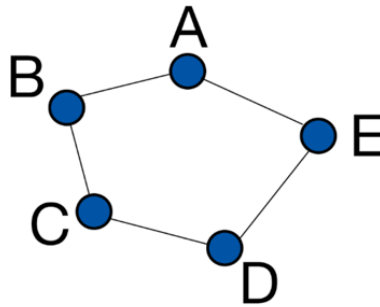
Notice also that in this world there is exactly one straight line between any two given points, just as in Euclidean Geometry.

World 2:

Let us complicate the world a little more and consider a world with a finite number of points such that every point has exactly two neighbors. The world will look like a necklace with beads.

Let us ask the same question. What do lines and straight lines look like in this world (we will assume the same definitions)?

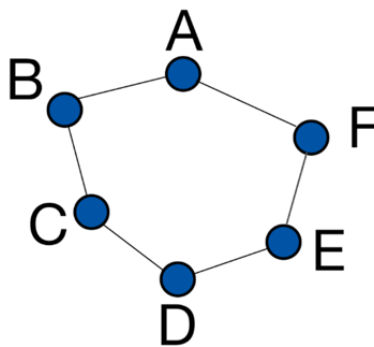
See the following example with 5 points:



Both ABCDE and AE are lines which connect A and E. So, in this world, unlike in the previous one, there can be multiple lines joining two points.

Are both of these straight lines? Clearly not. The length of ABCDE is 4 while the length of AE is 1. So, AE is a candidate for a straight line while ABCDE is not.

Can there be multiple straight lines between two given points? In the example above, that seems not to be the case. However, consider the following example:



Here, ABCD and AFED are distinct paths of length 3 connecting A to D. There are also no shorter paths. Hence, there are two straight lines between A and D. This is different from World 1 and from Euclidean Geometry.

Exercise: In the first example of World 2, there is exactly one straight line between two points, while that is not the case in the second example. Try to come up with a generalization for when it is the case that there is exactly one

straight line and when it is not. In order to do that, try worlds with 3 points, 4 points, 7 points and so on. See if you can figure out a pattern which will help you predict when there will be exactly one straight line between any two points and when there will be more than one.

Solution (In the form of a Dialogue): T: We have already seen that for World 2 with 5 points, there is exactly one straight line between two points. However, for World 2 with 6 points, there are certain pairs of points with more than one pair of straight lines between them. Can you figure out, for World 2 with 1807490242434 points, whether there are pairs of points with more than one straight line between them?

S: That seems impossible. I cannot draw those many points!

T: Okay. What a mathematician would try and do here would be to first generalize the problem. So, given World 2 with n points, are there pairs of points with more than one straight line between them? The next step would be to try some specific examples. We already have two examples – 5 and 6 points. Why don't we try some more? One thing before we do that. So far, we have only seen examples with pairs of points having one or two straight lines between them. Can pairs of points in World 2 have 3 straight lines between them?

S: No.

T: Why not?

S: To get from one point to another, there are only two ways to do so. We start at a point and then can either go clockwise or anti-clockwise. So, there are only two possible paths between two points.

T: Great! Let's get back to some examples. Try out World 2 with 3 points, 4 points, 7 points and 8 points. Tabulate your results.

S:

NUMBER OF POINTS	DOES EVERY PAIR OF POINTS HAVE A UNIQUE STRAIGHT LINE?
3	Yes
4	No
5	Yes
6	No
7	Yes
8	No

It seems like they alternate. So, I'm going to say that if n is even, there are

pairs of points with more than one straight line between them. If n is odd, that is not the case.

T: That is your conjecture. You have given what is the start of a scientific proof for the conjecture. You have collected a sample of outcomes and generalize from them. However, so far, we cannot be sure what would be the case with 1807490242434 points. In order to do that, we need a mathematical proof. Try to come up with reasons why your conjecture must be true.

S: How do I do that?

T: Think about your proof that there cannot be more than two straight lines between a pair of points.

S: I think I got it. Let the two points we are interested in be X and Y .

There are two paths between X and Y (clockwise and anti-clockwise). For there to be two straight lines, both these paths must be of equal length. That means there must be an equal number of edges on these paths (from our definition of length) That means there also must be an equal number of points on both these paths. Let the number of points on these paths be k . So, the total number of points in the World are $2k+2$ (k from each of the paths, plus X and Y), which is even.

T: Great! What you have shown is that ‘if there exists a pair of points for which there are two straight lines between them, then the number of points in the world is even.’ Notice, that this doesn’t mean that there do exist pairs of points for which there are two straight lines between them. The statement is only saying something about what happens if such a pair existed. You still need to prove that ‘if the number of points in the world is even, there exists pairs of points for which there are two straight lines between them.’ I will leave that to you as an exercise.

Extension Exercise: So far, all the worlds we have seen have either one or two straight lines between them. Attempt to create worlds where there are 3 straight lines between two points, 4 straight lines and so on. Try to come up with some rules governing these worlds.

A Note on Representation:

In World 1, just as in Euclidean Geometry, there is only one straight line between two points. So, we can represent that straight line by AB where A and B are the two points we are interested in. This is also true in World 2 with an odd number of points. However, in World 2 with an even number of points, as we showed, there may be more than one straight line between two points. In that case, if we wish to differentiate between these two straight lines, we need more

information than just the end points. We need at least one point on the way. For instance, in the world with 6 points, we can represent the straight lines between A and D by ABD (which is the same as ACD) and AFD (which is the same as AED). If we have an image drawn with an orientation, we could also call ABD, AD-a (a=anticlockwise), and we could call AFD, AD-c (c=clockwise). If we encounter worlds with more than two straight lines between pairs of points, we may need even more information in order to distinguish them.

Triangles and Other Polygons

Let us borrow the definition of triangle from Euclidean Geometry –

Definition 3: A triangle is a closed curve/line surrounded by 3 straight lines and nothing else.

So, notice that there are no triangles in the first world we studied above since there are no closed curves. However, let us take a look at the second world.

Let us see some examples of triangles. Recall World 2 with 5 points. 1. The 3 straight lines ABC, CDE, and EA form a triangle. 2. The lines ABCD, DE, and EA do not form a triangle since ABCD is not a straight line.

Let us recall a theorem from Euclidean Geometry:

Triangle Inequality: The sum of the lengths of two sides of a triangle is always strictly greater than the length of the third side.

Exercise: Does the triangle inequality hold in World 2? (Maybe it holds in some instances of world 2 and not in others)

Solution (As a dialogue):

T: Does the Triangle inequality hold in World 2?

S: It seems to work for World 2 with 5 points. The largest possible side in this world is of length 2 since the largest possible straight line is of length 2. The sum of the other two sides is 3 since the triangle must contain all points, which is greater than 2.

T: Great! That's good reasoning. What about World 2 with 6 points?

S: There might be a problem. Consider the straight lines ABCD, DEF and FA. They form a triangle. However, the length of ABCD is 3 and the sum of the lengths of DEF and FA is 3. So, in this world, the sum of the lengths of 2 sides

need not be strictly greater than the length of the third side, but I think it must be greater than or equal to the length of the third side.

T: Let us call that the ‘Weak Triangle Inequality’ (which states that the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side). Can you generalize what you have said about 5 and 6 points?

S: Maybe it has something to do with odd and even once again. My conjecture is: If n is odd then the triangle inequality holds. If n is even, the weak triangle inequality holds.

T: Can you give me reasons why that is true. Think about generalizing the argument you made for 5 points.

S: If n is odd, then we can write n as $2k+1$ where k is any whole number. Then, the longest possible side of a triangle is of length k . The sum of the lengths of the other two sides is $k+1$, which is bigger than k . So, clearly this will continue to be true if the longest side of the triangle has length less than k .

T: What about if n is even?

S: If n is even, then we can write it as $2k$. The longest possible side of a triangle is of length k . The sum of the other two sides is k . So, clearly this will continue to be true if the longest side of the triangle has length less than k .

T: Good job!

Exercise: Do equilateral triangles always exist in World 2? If not always, when do they exist? What about squares, equilateral pentagons, and other equilateral n -polygons (A n -polygon is an n -sided closed curve surrounded by n straight lines. A triangle is a 3-polygon, squares and rectangles are 4-polygons, and so on).

Solution (In the form of a dialogue)

T: So far, we have talked about general triangles. I want to switch to equilateral triangles. When do they exist in world 2. Does World 2 with 5 points have an equilateral triangle?

S: For something to be equilateral, all the sides must have equal length. Since a triangle has 3 sides, and the length of sides must be a whole number, the number of points must be divisible by 3. 5 isn’t so it cannot have an equilateral triangle.

T: Great! So, what you are saying is that ‘Equilateral triangles exist in World 2 if and only if the number of points is divisible by 3.’

S: Yes.

T: What about squares.

S: Squares also have equal length sides. There are 4 of them. So, the number of points, must be divisible by 4.

T: What about a general equilateral n -polygon. A n -polygon is an n -sided closed curve surrounded by n straight lines. A triangle is a 3-polygon, squares and in fact general quadrilaterals are 4-polygons, and so on. An equilateral one has sides of equal length.

S: Same as with triangles and squares. An equilateral n -polygon can only exist in a world where the number of points is divisible by n .

T: Nice. Let us go back to World 2 with 6 points. It has an equilateral triangle with sides ABC, CDE, and EFA. It also has an equilateral hexagon (6-polygon) with sides AB, BC, CD, DE, EF, and FA. Let m and n be two whole numbers. How many points are there in the smallest world which contains both an m -polygon and an n -polygon? Let us take an example: A 24-point world contains both an equilateral triangle (of side length 8) and a square (of side length 6). However, it is not the smallest such world. A 12-point world also contains an equilateral triangle (of side length 4) and a square (of side length 3).

S: 12 is 4 multiplied by 3. Maybe the smallest possible world is m multiplied by n ?

T: Think about the 6-point world.

S: Oh yeah! If my conjecture were true, the smallest world to contain a triangle and a hexagon would be 6 multiplied by 3 which is 18. Clearly, a world with 6 points contains both of these so my conjecture must be false.

T: Think about what you said earlier – A n -polygon exists in World 2 if and only if the number of points is divisible by n .

S: I got it. For both m and n polygons to exist in the world, the number of points must be divisible by both m and n . So, the number of points in the smallest world must be the smallest number divisible by both m and n .

T: That is sometimes called the Lowest Common Multiple (LCM). There are algorithms to calculate it for any two numbers. Well done! One last question. A world with 3 points contains an equilateral triangle but no other equilateral polygon. Are there other such worlds which contain exactly one type of equilateral polygons? I'm going to leave that one to you.

Circles

Definition: A circle with radius r and center O is the set of points at a distance r from O . The only restriction is that $r \geq 0$.

When we say distance between two points, we mean the length of the straight line path between those points.

Notice that in the worlds we are looking at, a circle need not be a closed curve. For instance, in the following example of world 1, the set consisting of A and C is a circle, with center B and radius 1, even though it is clearly not closed.



Exercise: What do circles look like in World 1?

Solution (In the form of a dialogue):

T: What are the possible circles in World 1?

S: Well, one point on its own can be a circle. For example, if the center is one of the end points, and the radius is 1, a circle can consist of one point.

T: Can there be 2 point circles?

S: Yes. In the diagram above, A and C constitute a circle of radius 1 with B as the center.

T: So, can any point be a circle?

S: I guess so. If we take the center to be one of the end points and the radius to be the distance from the end point to the point we want, that one point will be a circle.

T: Can any two points be a circle?

S: I think so.

T: Let us take two adjacent points. What is the center and radius of the circle which contains them?

S: I guess there isn't one. Maybe, any two non-adjacent points can be circles.

T: Let us try two points separated by two points (A and D in the diagram above)

S: That also cannot be a circle. Maybe two points can be a circle only if they are separated by an odd number of points.

T: That sounds reasonable. Try to think why that would be, but let's leave it at that for now.

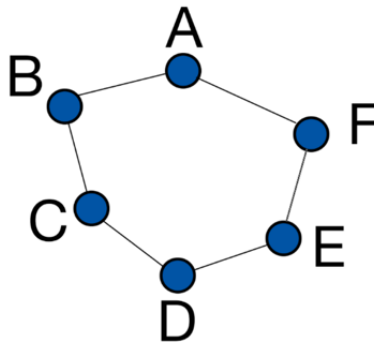
Exercise: What do circles look like in World 2?

Solution (In the form of a dialogue):

T: What do circles look like in World 2?

S: That seems much harder to answer.

T: Okay, let's take specific instances of World 2. Start with 6 points.



S: If we take A as the center. With radius 1, B and F constitute a circle. With radius 2, C and E constitute a circle. With radius 3, D constitutes a circle. With radius 4, once again, C and E constitute a circle.

T: Wait a second. Is there a circle with radius 4?

S: Yeah! You go 4 to the left from A and 4 to the right from A.

T: Give me the definition of a circle with center A and radius 4.

S: A circle with center A and radius 4 is the points which are a distance of 4 away from A.

T: Okay. What is distance?

S: I don't think we defined distance.

T: How would you define the distance between two points?

S: As the length of the path between those two points.

T: There might be many paths between two points. So, which path?

S: I guess the shortest path.

T: So, the distance between two points is the length of a straight line between them. Are there any points of distance 4 from A?

S: No! C is of distance 2 since the straight line is the anti-clockwise path from A to C.

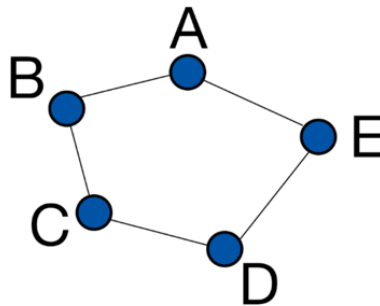
T: So, we only have circles of radius 1, 2 and 3. Let us get back to the problem of characterizing circles.

S: Clearly, any one point can be a circle if we take the opposite point as the center and radius as 3. Like in World 1, adjacent points or points separated by an even number of points cannot be circles. However, any point separated by an odd number of points can be a circle.

T: Great! What do you think is the generalization to any number of points?

S: Maybe it is exactly the same as in World 1. Any single point is a circle. Any two points are circles if they are separated by an odd number of points and they are not circles if they are adjacent or separated by an even number of points.

T: We should test that conjecture. Take the world with 5 points.



S: Oh! I don't think it works. I can't figure out a way to make single points into circles! Also, I think now that any two points can be a circle.

T: Great! However, don't give up hope on this. Maybe there is still a pattern, but a slightly different one to the one you found earlier. What you need to do now is to try more examples and see if you can spot something.

(The students should eventually come to realize that their original conjecture is true if and only if the number of points is even. For odd number of points all and only pairs of points are circles)

Extension Exercise: So far we have looked at circles in World 1 and 2. Think about them in the following possible generalizations of world 2:

1. A world with a finite number of points such that every point has exactly 3 neighbors
2. A world with a finite number of points such that every point has exactly 4 neighbors
3. A world with a finite number of points such that every point has exactly n neighbors

Extension Exercise: An ellipse in Euclidean geometry has two foci (the equivalent of center of a circle). It is the set of points such that the sum of the distances from a point to the two foci is constant. Clearly, a circle is an ellipse with the two foci being at the same point – the center. However, can you characterize non-circle ellipses in the worlds we have looked at so far?