

Geometry Education

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Geometry Education

In this article, I intend to give some understanding of the current state of research in Geometry Education, focusing on K-12 education. I will be using the notion of Geometry in the sense of Clements (2003) as:

“the study of spatial objects, relationships, and transformations; their mathematization and formalization; and the axiomatic mathematical systems which have been constructed to represent them.”

By Geometry Education, I mean the formal education system aimed at teaching Geometry. Geometry Education Research refers to the set of literature, including normative, empirical and theoretical research, which deals with Geometry Education.

While Geometry as the study of spatial objects, relationships and transformations has been an active area of inquiry for a very long time, the rest of Clements' definition is relatively new. In order to get an understanding of Geometry, Geometry Education, and Geometry Education Research, it is important to get some understanding of the changing conceptions of Geometry.

I will start this article by giving a quick history of geometry including the various conceptions of what geometry is across space and time. I will then move on to describing some general learning theories related to Geometry. Then, I will talk about how various educational bodies across the world, such as the Common Core and NCERT, talk about Geometry Education and its goals. I will end with research addressing more specific aspects of Geometry Education. Some of the studies discussed will have implications for teaching. Those will be highlighted.

History of Geometry

Geometry began in various parts of the world with measurement and relationships between different measurements such as length, angle, area, and volume. These relationships were determined empirically. Evidence for the existence of such geometry is found in Ancient Egypt, Babylon, China, Greece, and Vedic India (O'Connor & Robertson, n.d.; Lewy, 1949; Seidenberg, 1978). In most parts of the world, Geometry continued as a largely empirical science. For instance, as late as 7th Century India, Brahmagupta found the formula for cyclic quadrilaterals given the lengths of the sides (Weisstein, n.d.). However, he did not give a deductive proof for this claim. While these mathematicians were able to see geometric relationships and reason with them, they were not doing mathematics in the modern sense of the term.

The birth of what looks like modern mathematics appeared to have happened in Greece (Seidenberg, 1978). Euclid employed deductive proofs from axioms and definitions in order to arrive at conclusions. He was already familiar with many of the conclusions – his enterprise was to create a coherent system with a small number of starting principles from which the rest of the system could be deduced. He viewed Geometry as explaining the world, and the axioms as self-evident truths.

While the first four axioms appeared to be true of the world he saw around him, the Parallel Postulate did not seem to be self-evident. One of the main goal of mathematicians in the Euclidean tradition for the next millennia was to prove the parallel postulate from the other four axioms.

A significant effort was made by people such as the Arab Mathematicians al-Haytham and Omar Khayyam to prove the parallel postulate from a proof by contradiction (Katz, 1998).

This resulted in some interesting conclusions which would later be recognized as theorems in non-Euclidean geometries. It was only in the 19th Century that mathematicians started seriously exploring the alternatives to Euclidean Geometry. People like Lobachevsky and Bolyai (Bolyai and Lobochevsky, n.d.) were pioneers in this effort with their work in Acute Geometry, which was later developed into Hyperbolic Geometry by mathematicians like Riemann and Poincare (Cannon et al., 1997). Beltrami was the person who finally showed the independence of the Parallel Postulate from the other four (Eugenio Beltrami, n.d.).

The Twentieth Century saw Hilbert's rigorous axiomatization of Euclidean Geometry (Hilbert, 1992). It also saw the development of various different types of Geometry and Topology. However, areas such as Differential Geometry are not usually a part of K-12 education, and I will be ignoring those developments. There have also been some toy geometries developed in the last few centuries such as Paper-Folding Geometry (Rao, 1901) and Taxi-Cab Geometry (Krause, 1973) which have been used in K-12 education.

Theories and Frameworks of Geometry Learning

As mentioned in the introduction, I will be using Clements' notion of Geometry. The valuable thing about this definition is that it doesn't only involve the study of formal systems, but also involves getting an intuitive understanding of the objects of Geometry.

In this section, I will be presenting a few major theories and theoretical frameworks for Geometry Learning. A few of them are not specific to geometry. However, they have been widely used in the domain. There are many other theories which have been applied to geometry education (Sinclair et al., 2016) including the Framework of Spaces for Geometric Work (Gomez-Chacon & Kuzniak, 2015), Prototype Theory (Hershkowitz, 1990; Fujita, 2012; Yu,

Barrett & Presmeg, 2009), the Theory of Variation (Leung, 2008; Leung, 2014), and the Conception, Knowing, Concept (cK ϕ) model (Balacheff, 2013; Gonzalez & Herbst, 2009), which I will not be touching on.

Piaget and Inhelder's Theory

Piaget & Inhelder (2006) gave one of the first theories of how humans learn geometry.

There are two major themes in their conception (Clements, 2003):

1. Representations of space are constructed through the progressive organization of the student's motor and internalized action.
2. The progressive organization of geometric ideas follows a definite order that is more logical than historical. It starts with topology (continuity, etc.), followed by projective relations (rectilinearity) and finally Euclidean relations (parallelism, angle, distance, etc.).

As Clements & Battista (1992) point out, there is evidence for the first claim but not for the second. Children do seem to use their bodies and hands to learn about shape. However, the the ordering of understanding hypothesized by Piaget and Inhelder doesn't seem to exist. Children appear to have some Euclidean notions at a very early age.

Van Hiele Model

The van Hiele model of geometry learning, like Piaget and Inhelder's, also involves an ordered set of levels through which students progress in their learning of geometry. The model is based on the following assumptions (Crowley, 1987; Clements, 2003):

- Students proceed sequentially through levels, which cannot be skipped.
- Progress depends more on the content and instruction than on age.

- Concepts which students understanding implicitly at one level become explicit at a higher level.
- Each level has its own language and relations.
- If the instruction is at a different level from that of the learner, learning will not occur.

The model is sometimes described with four levels and sometimes with an additional 0 level (Clements, 2003). The levels of the model are as follows (Crowley, 1987; Clements, 2003; Jones, 1998):

1. (Level 0) Visualization: Students recognize geometric objects, but as wholes and not having components or properties
2. (Level 1) Analysis: Students begin to see the characteristics of shapes, and these properties help them create a classification
3. (Level 2) Informal Deduction: Students start seeing the relationships between properties, and informal arguments can be used to come to conclusions. Students can follow formal proofs, but not come up with them
4. (Level 3) Deduction: Students can prove theorems deductively and see the significance of deduction, and can see the distinction between necessary and sufficient conditions
5. (Level 4) Rigor: Students can work in a variety of axiomatic systems, allowing a higher level of abstraction.

A large amount of curriculum around the world has been based on this model (Clements, 2003). However, more recently, there has been a lot of criticism of the van Hiele levels, especially in relation to the nature and discreteness of the levels. Students have been found to

reason at various different levels at the same time, and go through the levels at different rates for different shapes (Burger & Shaughnessy, 1986; Clements, 2003).

Tall and Vinner's Concept Image & Concept Definition

Tall & Vinner (1981) first made the distinction between Concept Image and Concept Definition. A concept definition is the actual definition of the concept, while the concept image is the set of associated pictures, properties, and other images associated with the concept. This distinction is important in two ways. Firstly, it points to the fact that students need a rich concept image in order to work with the concept – a definition is not sufficient. Secondly, it could be the case that there is conflict between the Concept Image and Definition. This will come up again when discussing defining.

Fischbein's Theory of Conceptual Figures

Fischbein conceptualizes geometry as dealing with mental entities which possess both conceptual and figural aspects (Fischbein, 1993). Conceptual here means relating to a general class of objects based on their common features. Figural refers to our concrete realization of the concepts which include properties like shape, position, and magnitude.

Geometrical reasoning, then, is characterized by the interaction between these two aspects (Marriotti, 1995; Jones, 1998). These can come into conflict and, according to Fischbein, a large part of geometry learning is to do with working through these conflicts. This will come up again when discussing defining.

Realistic Mathematics Education

Realistic Mathematics Education (RME) is committed to giving students 'realistic' problems. 'Realistic' here refers to problems which are experientially real in the mind of the student rather than problems which are to do with the real world (Van den Heuvel-Panhuizen &

Drijvers, 2014, pp. 521-525). Within the RME tradition is a particular activity called Guided Reinvention. The idea behind this is that students create a body of knowledge, which being guided by the instructor, rather than being given that body of knowledge (Wubbels, Korthagen, & Broekman, 1997). Gravemeijer (1999) laid down a framework for activities within RME. It consists of four types of activities: Situational, Referential, General, and Formal. As Zandieh & Rasmussen (2010) put it, situational activities involve students working towards mathematical goals in experientially real settings. Referential activities involve models-of which refer, implicitly or explicitly, to physical and mental activities in the original task settings. General activities involve models-for which involve interpretations and solutions independent of the original task setting. Finally, formal activities involve students reasoning in ways which result in the creation of a new mathematical reality.

There is a lot of work in geometry in this tradition. One example at the high school level is Zandieh & Rasmussen (2010). I will discuss this further when talking about defining. Other examples I found, such as Dawkins (2015), deal with Geometry at the undergraduate level.

Duval's Cognitive Model of Geometrical Reasoning

Duval (1998 as cited in Jones, 1998) proposes that that geometrical reasoning consists of three types of cognitive processes, namely:

1. visualization processes such as the visual representation of a geometric statement
2. construction processes using tools
3. reasoning processes, particularly discursive processes, for the extension of knowledge, for explanation and proof

These three are related but can be performed separately. For example, visualization doesn't depend on construction. Also, even though visualization can help in finding a proof, it

can often be misleading. However, Duval points out that the synergy of these processes is important for proficiency in geometry.

Formal Educational Stipulations for Geometry Education

While discussing the educational literature, it is important to discuss the context in which that research is occurring, namely the stipulations for geometry education by educational bodies around the world. I will be using the examples of the US, India, Finland and Singapore. I picked the US and India since I have the most experience in these two educational systems. Finland and Singapore have been picked since they are regarded as having good mathematics curricula.

Common Core Educational Standards (USA)

The Common Core begins geometry in Kindergarten with students identifying, analyzing, comparing, and creating shapes. It move on to reasoning with and about shapes, and eventually to a more formal approach including proofs. The subheadings of the standards in K-8 (Common Core State Standards Initiative, n.d.-a) are:

- Identify and describe shapes (Kindergarten)
- Analyze, compare, create, and compose shapes (Kindergarten)
- Reason with shapes and their attribute (Grades 1-3)
- Draw and identify lines and angles, and classify shapes by properties of their lines and angles (Grade 4)
- Graph points on the coordinate plane to solve real-world and mathematical problems (Grade 5)
- Classify two-dimensional figures into categories based on their properties (Grade 5)

- Solve real-world and mathematical problems involving area, surface area, and volume (Grade 6)
- Draw construct, and describe geometrical figures and describe the relationships between them (Grade 7)
- Solve real-life and mathematical problems involving angle measure, area, surface area, and volume (Grade 7)
- Understand congruence and similarity using physical models, transparencies, or geometry software (Grade 8)
- Understand and apply the Pythagorean Theorem (Grade 8)
- Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres (Grade 8)

The High School Geometry course (Common Core State Standards Initiative, n.d.-b) involves the concepts of congruence, similarity, right triangles, trigonometry, circles, expressing geometric properties with equations, geometric measurement and dimension, and modeling with geometry. These involve constructions, proofs, and applications of theorems to specific situations.

National Council for Educational Research and Training (India)

The position paper on the Teaching of Mathematics by the National Focus Group deals with setting the basis for Mathematics Education in India, at least for those schools associated with NCERT (National Council of Educational Research and Training (India), 2006). At the Primary level, the paper states the importance of dealing with non-number areas of mathematics such as shape, spatial understanding, and a vocabulary of relational words which extend the child's understanding of space.

At the Upper Primary level, there is an emphasis on visualization and spatial reasoning, and also on students justifying conclusions at an informal level. At the Secondary stage, argumentation and proof become central to the curriculum, and geometry becomes more about reasoning with the shapes and space students already understand.

Finnish Curriculum

Finland has had significant success in various international mathematics tests like PISA. While there is not a lot stipulated centrally by an authority, and there is a lot of latitude for experimentation, there are a few key things students are expected to learn at various grade levels (Hemmi et al., 2017).

Grades 1-2 focus on perceiving the 3 dimensional environment, and noticing, naming and classifying figure. In grades 3-6, build, draw examine and classify objects and figures, they learn more about particular shapes. They are familiarized with scale, including enlargement and reduction. In grades 7-9, they learn more about particular shapes, understand similarity and congruence, and work with 3D figures.

In higher grades, students make observations and draw conclusions in geometry, they learn to solve practical problems using geometry, and they use technological tools in order to examine figures and solve application problems in geometry.

Singapore Curriculum

In Singapore, the mathematics curriculum at the primary level (grades 1-6) is based on a Problem Solving approach. The following represent the various aspects of their curriculum.

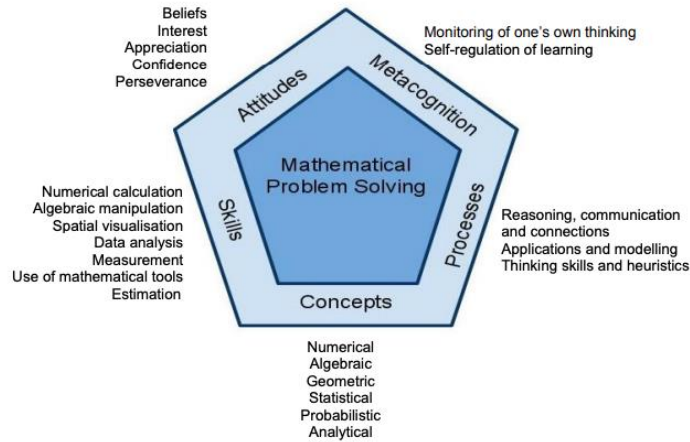


Figure 1 Singapore Curriculum

The Singapore curriculum also has detailed learning outcomes for each year for Geometry. Post the 6th Grade, Singapore has various options for students to pursue mathematics at various levels.

Research about Specific Aspects of Geometry Education

While all of these systems have differences in how they are formulated, there is broad consistency in the goals. Visualization and reasoning about shapes and space (spatial reasoning) and more formal reasoning are two of the common themes which go through all of these. The more formal reasoning involves definitions, proofs and axiomatic systems. I will now spend some time describing the literature in these areas.

I will also be talking about the role of non-Euclidean Geometries and of Technology in Education. There are other aspects of geometry, such as particular understandings related to particular objects and theories, coordinate geometry and its relationship to algebra, and the application of geometry outside of mathematics. I will not be focusing on these in this paper except for in relation to the other aspects mentioned above, given constraints of space.

Visualization & Spatial Reasoning

Unlike proving and axiomatic systems, research into visualization and spatial reasoning has clear implications right from elementary school. Being able to reason about, and using, shapes and space is clearly a valuable tool for students to have in order for them to navigate the physical world (Clements, 1998). Using maps, deciding on the size of a container for some object, creating data visualizations, and creating computer graphics are some of the practical applications of what is often referred to as ‘spatial reasoning’. Clements & Battista (1992) define spatial reasoning as:

the set of cognitive processes by which mental representations for spatial objects, relationships, and transformations are constructed and manipulated.

Kinach (2012) says:

Spatial thinking takes a variety of forms, including building and manipulating two and three dimensional objects; perceiving an object from different perspectives; and using diagrams, drawings, graphs, models, and other concrete means to explore, investigate, and understand abstract concepts such as algebraic formulas or models of the physical world.

Importance of Spatial Reasoning.

Hence, while spatial reasoning is not exactly the same thing as geometry, the two are closely related. Also, spatial reasoning is valuable far outside of geometry, including in other branches of mathematics and in other disciplines. In fact, research by cognitive scientists suggests that spatial reasoning plays an important role in predicting overall mathematics success, even more than scores in mathematics tests (Bruce & Hawes, 2015).

One important aspect of spatial reasoning is mental rotations of 2D and 3D shapes. Pazzaglia & Moe (2013) show that the ability to mentally rotate objects correlates with map reading skills. Tolar, Lederberg & Fletcher (2009) showed its correlation with achievement in the Scholastic Assessment in Mathematics (SAT-M). Kyttälä & Lehto (2008) showed that mental rotation ability had a correlation with scores in geometry, word problems, and mental arithmetic.

Recent research by Barbara Tversky (2019) suggests that spatial reasoning forms the ultimate foundations for reasoning and abstract thought. She claims that a lot of thinking involves the body. She even suggests that the concept of number, representations of numbers, and algebraic thinking and representations are grounded in spatial thinking.

Teaching and Learning of Spatial Reasoning.

While these studies indicate the importance of mental rotation and spatial reasoning more generally, they wouldn't have significant educational implications if these abilities were not trainable. However, as Uttal et al. (2013) shows in their meta-analysis, training is possible, and training transfers to tasks which have not been directly trained for. Their analysis also shows that the training is durable. The training they analyzed included video games, courses aiming at spatial reasoning generally, and spatial task training aiming at specific spatial skills. All of these appeared to have the desired effects.

I will now highlight a few examples of pieces of research to give a sense of the various teaching interventions related to Spatial Reasoning.

Bruce & Hawes (2014) showed that simple tasks of 2D and 3D mental rotation can have a significant impact on students' mental rotation abilities. This can happen with children even as young as 5. The tasks involved finding rotated versions of a given object, working with

interlocking cubes to create complex objects, and working on various other tasks involving orientation and position.

Symmetry is another important concept in spatial reasoning (Ng & Sinclair, 2015). Clements (1998) suggests that students start in Pre-K with creating shapes that have line symmetry. By the 2nd grade, they should be able to identify where the mirrors should be placed in order to break a shape into its symmetrical parts.

The learning of symmetry can be supplemented by tools such as rulers (Perrin-Glorian, Mathe, & Leclerc, 2013 as cited in Ng & Sinclair 2015) or by computer software (Battista, 2008; Clements, 2002; Edwards & Zaskis, 1993). The former can draw attention to specific aspects of symmetry such as the sides of a shape, while the latter has many benefits. Students using geometry software can see transformations occurring and view the symmetry. It can give students a dynamic view of symmetry (Ng & Sinclair, 2015). In their study with elementary school students, Ng & Sinclair (2015) showed that dynamic geometry environments along with mediation from a teacher can result in the creation of new language and gestures to communicate effectively about symmetry.

Mamolo et al. (2015) give a framework for the learning of various aspects of spatial reasoning. They suggest that this framework is useful at all levels. The framework consists of a network of concepts and tasks related to those concepts, which allows traversal in various different ways. Each node of the network consists of consists of a network of Key Developmental Understandings (KDUs), conceptual blending, and scaffolding. KDUs identify a qualitative shift in how students think about mathematical concepts and relationships (Mamolo et al., 2015; Simon, 2006). Conceptual blending refers to the bringing together of two representations and ways of reasoning.

Proof

In his book *How to Solve It* (1990), Polya, referring to a proof that the sum of the angles of a triangle is two right angles, writes:

If a student has gone through his mathematics classes without having really understood a few proofs like the foregoing one, he is entitled to address a scorching reproach to his school and to his teachers. In fact, we should distinguish between things of more and less importance. If the student failed to get acquainted with this or that particular geometric fact, he did not miss as much; he may have little use for such facts in later life. But if he failed to get acquainted with geometric proofs, he missed the best and simplest examples of evidence and he missed the best opportunity to acquire the idea of strict reasoning. (pg. 216-7)

High School Euclidean Geometry focusing on proofs begin in the United States in the late 19th Century (Herbst, 2002). The idea behind it was similar to that in Polya's quote above – to develop certain abilities in students rather than just focusing on knowledge transfer. Herbst shows how this ideal, due to practical considerations, eventually resulted in a disassociation between proving and knowledge construction with the advent of 2 column proofs.

As Weiss & Herbst (2015) point out, the current High School geometry course is often students' first introduction to proof, and sometimes their only example of proof in High School. Proof is rarely touched before High School. So, the discussion which follows will be constrained to that level. As many have pointed out, at least in the United States, the Geometry course is a caricature of actual mathematics, where form triumphs over substance, there are too many postulates, and there is a lack of clarity in the meanings of words (Christofferson, 1930; Usiskin, 1980; Weiss & Herbst, 2015; Weiss, Herbst, & Chen, 2009).

There are many concepts of what constitutes a Proof in the literature (Weber, 2014). Harel & Sowder (1998) introduce the notion of a Proof Scheme, i.e., the criteria by which an argument is considered convincing. They suggest that there are broadly three proof schemes: authoritative, empirical and deductive. Within deductive proofs, there are symbolic, formal proofs and axiomatic deductive proofs. Within the Proof Scheme framework, a mathematical proof is one which is convincing to mathematicians.

Weber & Alcock (2009) defined proofs as arguments within a representational system. Stylianides (2007) requires that proofs should be deductive and in an age-appropriate representation system. Hence, a proof is only one when it is understandable to the audience it is intended for.

Weber (2014) argues that all of these definitions are incomplete and too specific of since they are neither necessary nor sufficient in practice. He suggests an alternative conception of proof which could be thought of as a theory of proof. In this conception, the notion of proof is characterized and constrained by certain propositions, which are a combination of the conceptions mentioned above. An argument which satisfies all of those is definitely a proof, while an argument which doesn't satisfy any is not. Things in between such as picture proofs in geometry and topology are controversial.

It is also important to point out here that, as Weber et al. (2014) says, mathematicians themselves do not necessarily gain conviction about results through deductive proofs. In fact, many mathematicians will often use authority and/or empirical means to get convinced about results before using them since it would be impossible for them to work through detailed proofs for each result. Hence, the goal of proof education need not necessarily be that students prove every result, but that they see the value of deductive proof.

Before moving into curriculum and teacher implications of research, it would be useful to look at how students understand proof.

Student Understanding of Proof.

Schoenfield (1986) shows a surprising result regarding students understanding of proof after their geometry course – students saw empirical methods as determining truth, while deductive arguments were just exercises teachers gave them. Knuth & Elliot (1998) found that even those students who would be considered sophisticated mathematically gave empirical justifications. This has been shown again and again in various studies such as Harel & Sowder (2007) and Martinez & Pedemonte (2014).

In another study by Martin & Harel (1989), fifty two percent of student teachers accepted an incorrect deductive argument as a proof for an unfamiliar statement. Fischbein & Kedem (1982) and Harel & Sowder (2007) found that even after accepting a deductive argument, high school students saw room for potential counter-examples, while Galbraith (1981) found that over a third of the students they studied did not understand the concept of counter-examples and 18% of them thought a single counter-example was insufficient to disprove a claim (Battista & Clements, 1995).

Harel & Sowder (1998) found something similar when working with University students. They did not have an axiomatic proof scheme, and relied on empirical and authoritative proofs. Even when reasoning deductively, students make inferences which do not follow from their premises. For instance, they conclude $Q \rightarrow P$ from $P \rightarrow Q$ (Sowder & Harel, 2003). Students also struggle to read proofs and to judge their suitability (Hoyles & Healy, 2007; Lin & Yang, 2007).

As mentioned above, students tend to believe that the reason you prove is in order to complete a task assigned by a teacher, or to verify something that you already know to be true (de Villiers, 1995) – there is no discovery associated with proof.

I want to highlight two recent studies which attempt to construct models of student understanding of proof. The first is a model by Ahmadpour et al. (2019) shown in Figure 2.

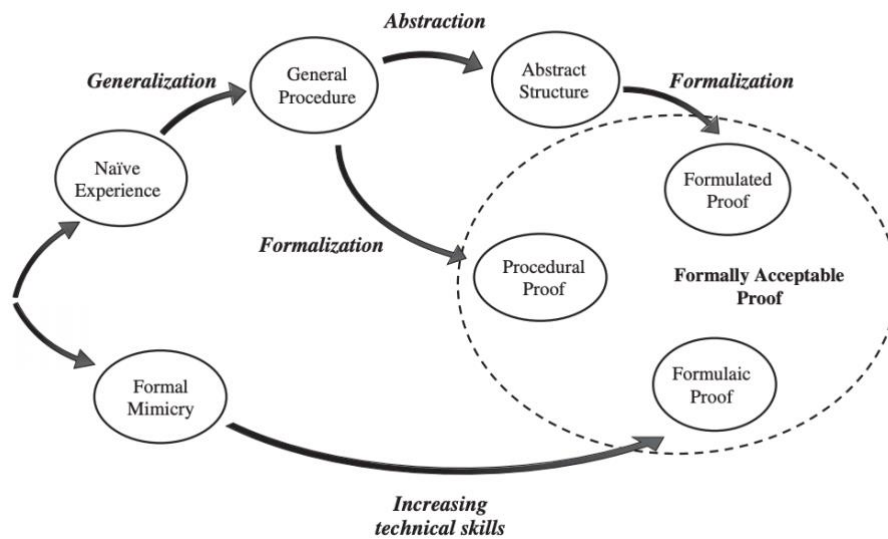


Figure 2 Students ways of understanding a proof

The ovals represent states of understanding while the arrows represent transitions between states. The three ovals in the larger oval are possible end states of a students’ understanding of proof. Students can move forward or backward in this path, and can often even switch paths depending on context.

Another model, by Miyazaki et al. (2017) is shown in Figure 3.

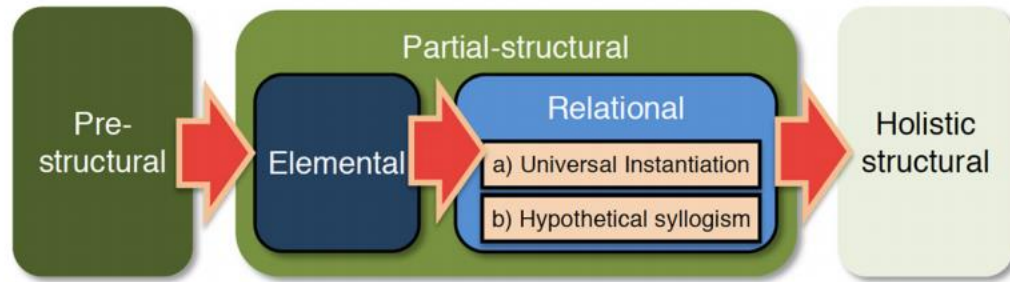


Figure 3 Framework for the understanding of deductive proofs.

At the pre-structural level, students see proof as a cluster of symbolic objects empty of meaning. At this stage, students would answer a question such as “in order to prove that the base angles of an isosceles triangle ABC are equal, what theorems are needed in order to deduce $\triangle ABD \equiv \triangle ACD$?” (D is the midpoint of the side BC), with something a random singular proposition such as “ $BD=CD$?”.

At the partial-structural level, students need to recognize the elements of a proof along with some logical chaining relationships between the components. This involves an understanding of the distinction between premises and conclusions, and then understand two types of relationships which construct the chaining relationships, which they call Hypothetical Syllogism and Universal Instantiation.

In the final Holistic-Structural level, students are able to reconstruct previously taught proofs, but also plan and construct their own proofs. They also begin to understand the relationships between theorems.

Implications of Research for Proof Education.

Given that students conceive of proof as verification or as doing an assigned task, it is clear that proof education is severely lacking. There have been efforts to improve this over the years. Polya and Fawcett are two early examples of that. Polya conceived of proof as problem solving (Polya, 1990). What he focused on were techniques and heuristics in order to conjecture

and prove such as using empirical means to come up with conjectures, generalizing conjectures in order to prove them more easily, and so on.

Fawcett (1938) provides an extremely interesting example of mathematics education research. The research touches on things beyond proof, and I will return to this work in other sections as well. Fawcett created a course which was aimed at students constructing Euclidean Geometry. Students engaged, not just in coming up with conjectures and proofs, but also in laying out the axiomatic system on which their proofs were based. The results of the experiment seem to have been universally positive. Not just did students learn how to prove, but they also learnt the same amount of geometry that other students knew, as was demonstrated through tests after the experiment. The students who went through the course judged it to have had a significant impact on their lives many years after the fact (Flener, 2009).

While both Polya and Fawcett achieved seeming success, the impact of their work on the actual Geometry course has been minimal (Herbst, 2002). More recent research into proof has seen a reconceptualization of the notion of proof, as mentioned above. Proof as a convincing argument, and the introduction of levels of proof and proof schemes give new tools to proof educators. Rather than thinking of students as being misconceived, these concepts allow us to locate students' thoughts, reasoning, and motivation.

Moving on to some specific areas of interest related to proof production in Geometry, generating examples of concepts has been hypothesized to help in proof production (Weber et al., 2008; Watson & Mason, 2005). However, there has not been any significant effect found to this practice – indeed, it seems to be as effective as students studying given examples (Iannone, Inglis, Mejia-Ramos, Simpson, & Weber, 2011). However, in the same study, they do point out that the methodology of example generation might affect their result.

There has also been some research on empirical verification after proof, especially in the case of tasks name ‘proof problems with diagrams’ (Komatsu & Tsujiyama, 2013; Komatsu, 2017), where statements are made in reference to a diagram. In such cases, a statement may be false or may require additional specifications since the diagram may contain certain hidden assumptions not made in the original statement. It also may be the case that examining examples after a proof may allow one to prove something more general. Komatsu (2017) presents the following diagram to aid in task design:

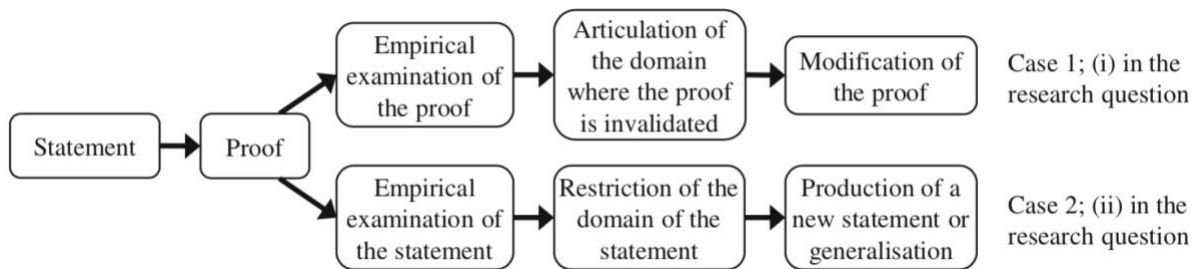


Figure 4 Model for Empirical testing after proof

The paper also suggests the following roles of the teacher:

1. Prompting students to draw diagrams different from the given diagram, or presenting such diagrams
2. Posing questions which move students to either revise the statement or proof
3. Selecting students with ideas worth examining and getting them to present to the class

Dealing with another class of proofs, namely existence proofs, Samper et al. (2016) suggests that such proofs are not intuitive. Usually, a student’s move is to impose the conditions on a randomly chosen object. The paper highlights the need for the teacher to play a role in mediation, and suggests that we cannot expect students to work through such proofs

autonomously without the help of a ‘more competent doer,’ a teacher who has worked through the details themselves.

Definitions

Closely related to proof is definition. As DeVilliers (1998) puts it, there are two ways of dealing with definitions: to teach definitions or to teach students to define. The latter does not imply that students ought to come up with every definition. However, learning to define is an important ability which students ought to learn. Freudenthal (1973) goes on to say that instructors are denying a learning opportunity for students by giving them definitions (as cited in Edwards & Ward, 2008).

A Good Definition.

What constitutes a good definition in mathematics? Edwards & Ward (2008), drawing upon work from van Dormolen & Zaslavsky (2003), give two sets of criteria for good mathematical definitions: necessary criteria and preferred criteria. The preferred criteria include minimality, elegance, and exclusion of degenerate cases. The necessary criteria come largely from Aristotle:

1. Hierarchy: Objects must be special cases of other objects
2. Existence: The object must be instantiated at least once
3. Equivalence: Multiple definitions must be shown to be equivalent
4. Acclimatization: The definition must fit into a deductive system

Definitions and Concepts.

As Vergnaud (1991) points out, a definition on its own will not enable a learner to apprehend and comprehend a concept (as cited in Ouvrier-Buffer, 2006). Rather, situations and problem solving give a concept meaning (Ouvrier-Buffer, 2006). Hence, the construction of

definitions requires students to play with a concept in various situations, extract the important aspects of that concept, and work with various representations of that concept before defining.

Defining in geometry involves many different aspects. Unlike in other areas of mathematics, students have access to mental pictures and diagrams of the objects they are defining. However, this can also make things harder. Mariotti and Fishbein (1997) talk about harmonizing the conceptual and figural aspects of geometric objects. The paradigmatic example of this is that of a square and rectangle. Students, especially in elementary school (Bussi & Baccaglini-Frank, 2015; Kaur, 2015; Tsamir et. al., 2015), tend to see squares and rectangles as different entities rather than squares being a subset of the class of rectangles. There is a conflict between their perceptual experiences, the figural aspects, and the need to unify and generalize, the conceptual aspects (Bussi & Baccaglini-Frank, 2015; Mariotti & Fischbein, 1997).

Implications of Research for Teaching Definitions.

Tall & Vinner's distinction between Concept Image and the Concept Definition (Tall & Vinner, 1981) has had a significant impact on the field of Mathematics Education Research. The Concept Image is the set of pictures, representations, properties and statements associated with concept in the learner's mind. The Concept Definition is the actual definition of the object. The biggest impact is that it has shown that learning the definition is not enough – students need an understanding of the object the definition is referring to.

Zandieh & Rassmussen (2010) use the Concept Image-Definition framework along with Gravemeijer's (1999) RME activity framework described in an earlier section to create a framework for Definition as a Mathematical Activity (DMA). They use this framework to construct a series of activities which transition students from triangles on the plane to creating

and inquiring into the concept of spherical triangles. They use students understanding of flat triangles and get them to define spherical triangles which keep the same essence.

Mariotti & Fischbein (1997) talk about two types of definitions: the basic objects of the theory and new elements within the theory defined in terms of the basic objects. These basic entities have a close relationship with the axioms of the theory. Using this and Fischbein's conceptual-figural distinction discussed above, the paper proposes a pedagogy for coming up with definitions via a classroom discussion. This involves:

- Observing
- Identifying the main characteristics
- Stating properties based on them
- Returning to observations to check

In a teaching experiment on defining and classifying quadrilaterals, Fujita, Doney, & Wegerif (2019) found that through a semiotic/dialogic process, students were able to transform their intuitions of what parallelograms were to an collective notion. Even though this notion did not necessarily agree with the conventional definition, students were able to use their definitions to solve other problems.

Axiomatic Systems/Theories

Since definitions and proofs only make sense within a theory/axiomatic system, it is important to touch upon education research into mathematical theories. Historically, there are broadly two conceptions of a mathematical theory, what Feynman (1965) calls the 'Babylonian' and the 'Greek' traditions. The Babylonian tradition conceives of mathematical theories as an interrelated network of facts, where you can derive many of the things you forget from things you know. The Greek tradition is axiomatic – there is small set of things we assume and we

derive a body of knowledge from that set. Formally, modern mathematics is in the Greek tradition. What both these traditions have in common is the commitment to relationship between statements of the theory.

Gowers (2000) set out a distinction between Problem Solving and Theory Building in mathematics. The distinction he made was a matter of priorities. Problem Solvers understand mathematics to solve problems while Theory Builders solve problems to understand mathematics better.

As Bass (2017) points out, there is not really much research on students constructing mathematical theories, especially before the undergraduate level. One exception is Fawcett (1938) who documents possibly the first attempt at the creation of a systematic course aimed at theory building. He starts with high school students listing geometric objects. Over the length of the course, this gets translated to a theory of space with definitions, undefined entities, axioms, and theorems.

Other Types of Geometry

Most of the examples given above are related to flat, gradient, 2D geometry. However, there has been some work on other types of geometry in the research literature. In this section, I will focus on 3D geometry, spherical geometry and discrete geometries. There are also other types of geometry such as projective geometry and paper folding geometry, which have been mentioned in the literature.

3D Geometry.

Sarfaty & Patkin (2013) found that Elementary school students are able to identify 3D shapes in their 'typical positions', but found it much harder to identify the same solid in different positions (Sinclair et al., 2016). Dynamic Geometry Environments (DGEs) give a new way to

work with 3D objects. The ability to drag and rotate makes it easier for students to get an understanding of these shapes (Leung, 2011; Sinclair et al., 2016).

An interesting aspect of flat 3D geometry is the results which can be extended from 2D flat geometry. This includes results about centroids, medians, perpendicular bisectors, circumcenters, etc. (Mammana et al., 2009; Sinclair et al., 2016). There are also results like Varignon's Theorem, that the midpoints of the sides of an arbitrary quadrilateral form a parallelogram, and Viviani's Theorem, that the sum of the distances from an interior point to the sides of an equilateral triangle is constant, which generalize (Sinclair et al., 2016). The extension of these results can be explored through DGEs.

Spherical Geometry.

Spherical Geometry refers to a 2D geometry where the surface we are dealing with is a sphere. We are concerned with similar things as we would be in Euclidean Geometry, but we are constrained to the surface of a sphere. As mentioned at the beginning, the advent of Spherical and other geometries resulted in a significant change in the nature of mathematics.

Lenart (2003), making the case for teaching different types of geometry alongside Euclidean Geometry, suggests that Spherical Geometry is especially useful since the sphere shape is prevalent naturally, most obviously in the shape of the Earth (Sinclair et al., 2016).

Junius (2008) worked with students, moving them from an extrinsic view of straightness of lines to an intrinsic view. It requires us to take the intrinsic view to see great circles as equivalent to straight lines.

Zandieh & Rassmussen (2010), as discussed in a previous section, worked on the notion of spherical triangles with students. They introduced the concept of a spherical triangle in stages, starting with planar triangles.

While there is some research in the area of Spherical Geometry for K-12 students, I have been unable to find much on other Elliptic Geometries, or on Hyperbolic Geometry.

Discrete Geometries.

Unlike Spherical Geometry, Discrete Geometry doesn't refer to a single axiomatic system. Rather, it refers to a collection of geometries which are non-gradient. Some of these geometries have points with internal structure, such as geometry related to simplices or some pixel geometries. Others, such as geometric graph theory, do not. Taxi-cab geometry is one such geometry which has been used at the K-12 level (Ada & Kurtulus, 2012).

Technology in Geometry Education

While I have intermittently talked about technology in before this point, given the focus on it in the literature, it is worth having a short section focused on the topic.

Various people have talked about the need for students being given the opportunity to explore (Polya, 1963; de Villiers, 2010). Virtual environments allow for that in ways which were unfeasible earlier. The ability to drag and rotate objects allows students to understand them much better (Hoyles & Jones, 1998; Soldano, Luz, Arzarello, et al., 2019; Laborde, 2005).

Soldano, Luz, Arzarello, et al. (2019), through a game which involved moving objects around in order to achieve some goals such as constructing parallelograms, found that students seemed to develop certain forms of strategic reasoning helping them discover, refute and verify conjectures. Gol Tabaghi & Sinclair (2013) showed that Dynamical Geometry Environments can support students in their synthetic-geometric thinking, which refers to thinking about objects in space.

However, there are various issues with interface design, and various tradeoffs in different virtual geometry environments between things like functionality and complexity (Mackrell, 2011; Sinclair et al., 2016).

Conclusion

In this paper, I have attempted to give an overview of the state of research on some aspects of geometry education of the last few decades. If there has been any shift in the nature of the research, it has been in the importance given to student conceptions and understanding. The clearest example of this is the case of proof. Early research was on instructional design. This moved to research on students conceptions of proof, but only looking for their deviation from the norm. It is only since the 1990s that serious attention has been given to actual students' conceptions, and not treating them as deviations.

There has been a large amount of research in areas like proof, defining, and spatial reasoning. However, there are many unanswered questions. For instance, we still don't have good ways to transition students to seeing value in a deductive proof scheme. There is also a lot we don't know about sensitizing students to the nature of a good definition.

There is also a lot we don't know about curricular sequencing. Piaget's idea of age based sequencing and the van Hiele's level based sequencing are two ideas we currently have, both of which have received significant critiques.

There are other areas of geometry, such as non-Euclidean geometries and theory building, which are only just entering the field in a significant manner. Given that these areas are in their infancy, there is a lot of work left to do.

Advances in technology will constantly give us newer tools with which we can explore geometry. Virtual reality has not been used very widely yet in the field. However, once prices drop, that will probably change.

That is not to say that earlier research isn't valuable. Fawcett's work (Fawcett, 1938) is still one of the most interesting ideas for geometry education. Similarly, Polya's work (Polya, 1963; 1978; 1979; 1990) has many insights which we should be making use of.

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