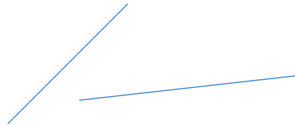Madhav Kaushish

# Parallel Lines

## Learning Outcomes

- Defining Objects Precisely
- Comparing and Evaluating Definitions
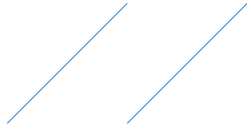
T: What are parallel lines?

S: They are lines which do not meet.

T: Are these lines parallel?



S: No. They are segments, not lines. Lines go on forever.

T: So are you saying segments cannot be parallel. Are the two segments below parallel?



S: Yes they are. How about if we say that 2 lines or segments are parallel if they do not meet even when extended.

T: Okay. How about the following. Are they parallel lines?



S: No, they are curves, not lines.

T: I'm guessing that when you say line, you mean straight line and when you say curve, you mean a non-straight path. Is that right?

S: Yes.

T: The words are a little confusing since we seem to have two commonly used terms for the same thing: line and straight line. Just for the purposes of this session, let us use the following classification:

Lines ⟶ Straight Lines

Lines ⟶ Non-Straight Lines
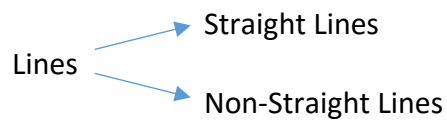
A line is something you can trace with your finger without lifting it (of course in the geometry we are working in, you cannot actually trace a line since it has no width). It could be finite or not. We can call segments finite lines.

S: Does that mean that when you ask us for what parallel lines are, you want us to include non-straight lines?

T: That is your decision. We would like the definition to be as general as possible. So, if you can come up with aa definition which covers both straight and non-straight lines and leads to interesting theorems, then you should do that. If you think concentrating on straight lines is better, you can do that. In order to make that decision, let us put down some common alternative definitions of parallel lines along with some scenarios:

Definition 1: Two lines are parallel if they do not meet.
Definition 1': Two lines are parallel if they do not meet when extended.
Definition 2: Two lines are parallel if they are equidistant.
Definition 3: Two lines are parallel if there exists a third line which is perpendicular to both.

S: How do we evaluate these.

T: Notice that Definition 1 is useless given the scenario we already put forward above. By Definition 1, any two paths would be parallel as long as they don't share a point in common.

S: What about Definition 1'. What does it mean to extend a non-straight line? See the following. How would we extend it?



T: Great observation. Unless we can come up with some way of extending any line, Definition 1' only makes sense for straight lines since it is obvious how we can extend straight lines.

S: What about definition 2?

T: What exactly do we mean by equidistant? This is quite a confusing notion for non-straight line, so let us put it aside for now. Let's try Definition 3. To illustrate definition 3, let us see what it means for straight lines:

C
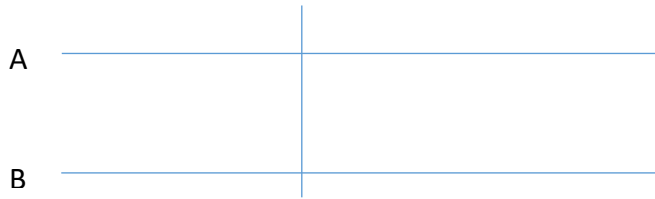
S: This seems to work for straight lines. What does it mean for a line to be perpendicular to a curved line.

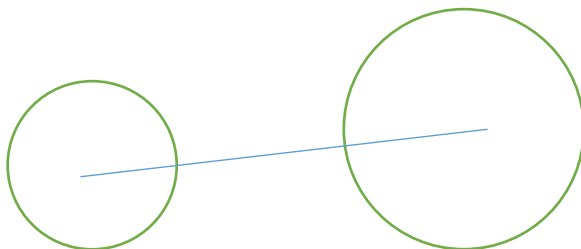T: Does it work for straight lines? Let me give you a scenario:
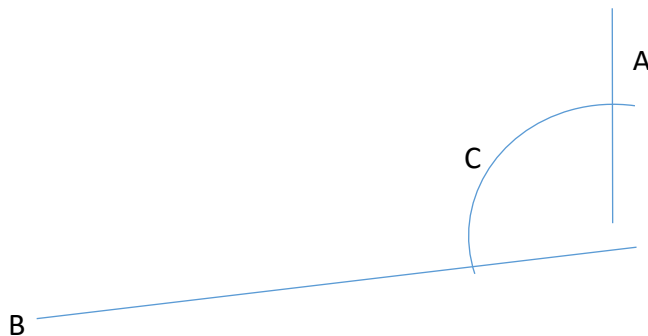


Both A and B are perpendicular to C. However, would you judge A and B to be parallel.

S: No. But, we can fix the definition to say:
Definition 3': Two lines are parallel if there exists a **straight** line which is perpendicular to both.

T: Let me illustrate a consequence of that:



If we draw a straight line connecting the center of any two circles, it is perpendicular to both. Hence, any two circles are perpendicular.

S: How is the line connecting the centers perpendicular to the circle? What does it mean for a straight line to be perpendicular to a circle?

T: Good point. In order to understand this definition, we need to understand what it means for two lines to be perpendicular. To start off with, what does it mean for two straight lines to be perpendicular?

S: They are perpendicular if there is a 90 degree angle between them.

T: 90 degrees is just a matter of a measurement tool. You can measure lines in the sort of geometry we are doing, right? You probably heard that lines have no width, so how can you measure them? Also, Napoleon used a 100 degree measure rather than a 360 degree measure. If he had won, we would probably be using that now. In that case, what we now call 90 degrees would be 25 degrees. Can you come up with a definition which doesn't require us to measure?

S: Can't think of one.

T: Let me give one and you can see if it works. We start with saying that two lines are perpendicular if they are at right angles. Now, we have to define right angle. We say a right angle is a quarter of a full rotation.

S: That makes sense. In the measurement as well, 90 degrees is a quarter of 360 degrees. However, it still doesn't make sense for the circle stuff.

T: How about the following: Line A is perpendicular to line B if the portion of line B on one side of line A is a reflection of the portion on the other side. By that, what I mean is put a mirror on line A. If what you get when you look through the mirror is exactly line B, then line A is perpendicular to line B.

S: What happens if A is curved?

T: That's interesting. It doesn't seem to make sense if A is curved. So, for now let us say that A has to be straight. Later, you might want to explore a scenario where A is curved.

S: So, a consequence of that is the following are perpendicular:
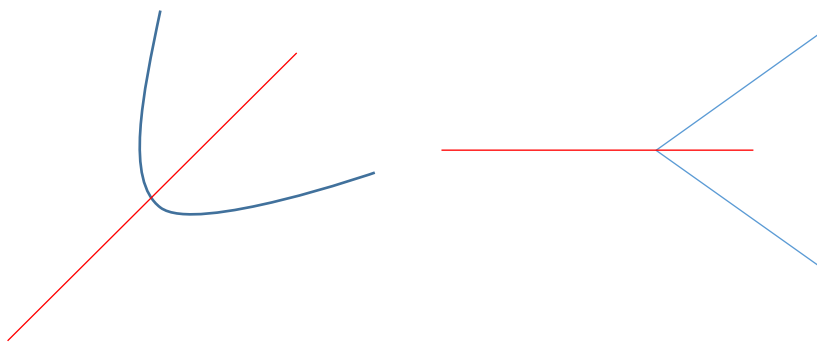
T: You're right. Let us accept this for now. If we get back to the circles, are you okay with accepting that they are parallel?

S: It seems quite weird.

T: It is worth exploring parallelness of curves more carefully. However, for now, let us stick to straight lines. Before we move on, as an interesting aside, notice that our definition of perpendicular allows for line A to be perpendicular to line B while line B isn't perpendicular to line A. For straight lines, it seems obvious that A perpendicular to B implies B perpendicular to A. Can you come up with a proof for that?

Moving on, so far we have been looking at parallel lines on flat surfaces. Let us move from flat surfaces to curved surfaces, specifically spheres. What do you think parallel lines on spheres look like? Think about globes of the Earth you may have seen.

S: Well, latitudes look like they are parallel. They are equidistant from each other. They also do not meet and I think any longitude would be parallel to them by our definition.

T: How about if we restrict to straight lines on a sphere? Are latitudes straight lines?

S: No, they are circles.

T: They might be circles, but are they straight lines?

S: How can a circle be a straight line?

T: What is a straight line?

S: What we learnt in class was that a straight line is the shortest path.

T: Take a model of the globe and take a string. Pick any two points on the same latitude (pick them in the same hemisphere but not too close and not the equator). Make sure the string is going through the two points and tighten the string so that it gets as short as it can be. You will notice that the string does not go through the latitude. However, if you try longitudes, you will see that they are the shortest paths. In fact, not just longitudes but any great circle is a straight line on a sphere. A great circle is a line which connects any two antipodal (opposite) points on a sphere. So, what is the consequence if we restrict our definitions of parallel lines to these 'straight lines'. Are there parallel lines?

S: Well, all great circles meet so by that definition they are not parallel. They are not equidistant either so they fail by that definition as well.

T: However, if you take the third definition, then any pair of great circles are parallel. To see that look at them drawn on a globe. You will notice that they intersect at two antipodal points. Think of these as the 'North' and 'South' Poles. Then there is a great circle which is the equivalent of the equator. We can draw that on the globe. You can see that it is perpendicular to both the great circles we were interested in.

S: So, the definitions resulted in the same consequences for straight lines on a flat surface but not on a sphere.

T: Exactly! That is why you need to be careful with checking definitions. That is also why you should always try to use a definition in areas where it originally wasn't intended – it can have interesting consequences. I will leave you with a few questions:

1. On a flat surface, the sum of angles of a triangle is two right angles (180 degrees). What is the sum of the angles of a triangle on a sphere? Remember a triangle has to be surrounded by straight lines. Think about other shapes as well.

You had remarked earlier that 'circles can't be straight lines.' Great circles are straight lines by your own definition. The interesting question is 'are they circles?' If they are, then circles can be straight lines. What about latitudes? Are they circles on a sphere? In order to answer that you need to define what a circle is. Remember that when we are working on a sphere, we want to stay within the sphere. So, the center of the circle cannot be inside the sphere. It must be on the surface.