Representing Numbers

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Learning Outcomes

1. Coming up with a solution to a specific problem
2. Generalising the solution
3. Extending the solution outside its domain of applicability
4. Proving the solution is the 'best possible' solution (for 10th graders onwards)
5. Understanding of various number representations

Task 1

You are given one each of 5 distinct objects: a fork, a spoon, a knife, a tennis ball, and a marble. You have to give each of these objects numerical values such that you can create the most number of consecutive numbers beginning with 1. For example, consider the following values:

Fork = 1

Spoon = 2

Knife = 3

Tennis ball = 4

Marble = 12

With the above values, I can create the following consecutive numbers:

1 = fork

2 = spoon

3 = knife

4 = tennis ball

5 = knife + spoon

6 = tennis ball + spoon

7 = tennis ball + knife

8 = tennis ball + knife + fork

9 = tennis ball + knife + spoon

10 = fork + spoon + knife + tennis ball

I cannot create 11 in any way, so even though I can create 12, the most number of consecutive numbers I can create beginning with 1 is 10. Can you do better than 10? Try to find the optimal solution.

Student: Clearly, if we replace the value of the Marble with 11, we should be able to go much further

Teacher: How much further?

Student: We could try it out. But, with all of these combinations, we might make a mistake!

Teacher: To reduce the risk of a mistake, try and put an upper bound on the number of consecutive numbers.

Student: What is an upper bound?

Teacher: An upper bound here is a number beyond which we know you will not be able to proceed. However, you might not be able to get to that upper bound. To give you an example, think about a simple situation with only a fork and a spoon with values 1 and 2 respectively. No matter how we combine them, we will never be able to go beyond 10, right?

Student: Yes. In fact, you will not even be able to go beyond 3!

Teacher: Why do you say that?

Student: Well, we know that the only way we can create numbers is by adding the values of objects together. Since we only have 1 fork and 1 spoon, we will not be able to go past their sum, which is 3.

Teacher: Good job! Now, can you say something about 5 objects with the fork, spoon, knife and tennis ball with values as above, but the marble having a value of 11?

Student: Once again, we wont be able to go past their sum, which you will have to give me some time to calculate

Teacher: Use my calculator. Many professional mathematicians cant do basic arithmetic. If you wish to acquire mathematical abilities, concentrate on thinking rather than on doing things that a computer can do!

Student: Okay, got it. An upper bound for the consecutive numbers is 21.

Teacher: As I mentioned earlier, an upper bound is not necessarily reachable. Now, you have to show that you will be able to reach 21.

Student: Lets try out all the numbers till 21

1 = fork

2 = spoon

3 = knife

4 = tennis ball

5 = knife + spoon

6 = tennis ball + spoon

7 = tennis ball + knife

8 = tennis ball + knife + fork

9 = tennis ball + knife + spoon

10 = fork + spoon + knife + tennis ball

11 = marble

12 = marble + fork

13 = marble + spoon

14 = marble +knife

15 = marble + tennis ball

16 = marble + knife + spoon

17 = marble + tennis ball + spoon

18 = marble + tennis ball + knife

19 = marble + tennis ball + knife + fork

20 = marble + tennis ball + knife + spoon

21 = marble + fork + spoon + knife + tennis ball

Student: So, we can get to 21. This means that we can't go beyond 21 with this combination, so 21, as we said, is an upper bound.

Teacher: When an upper bound is reachable, it is known as the least upper bound. This means that there is no upper bound lesser than 21. Nice! Can you do better than 21 with a different set of values for the objects?

Student: Hmm… How should we proceed?

Teacher: Well, see what is happening. 5 can be made in two ways with the current values. 5 = knife + spoon, but 5 = tennis ball + fork as well. So, there is some 'wastage' here. There are a limited number of combinations which can be made with 5 objects. If there are various combinations which give the same value, it stands to reason that you will not be able to get very far. Try and minimise wastage.

Student: We know we need 1.

Teacher: Why do we need 1?

Student: The question requires us the create 1. Since 1 is the smallest positive number, it cannot be made by adding together any other positive numbers.

Teacher: Okay. Let the fork be 1

Student: Now, we need to create the number 2. So, there are 2 ways we could do this. We could either give the spoon a value of 1 or of 2. However, since we wish to minimise wastage, we shouldn't give it a value 1, rather a value of 2.

Teacher: Okay, so what about creating 3?

Student: Hmm… Ah! you can do that by combining the fork and the spoon!

Teacher: Then, creating 4?

Student: We can't do that with what we currently have. So, lets give the knife a value of 4?

Teacher: Why not a value of 3 or 5?

Student: We cant give it a value 5, since we won't be able to create 4 with it. Giving a value of 3 will result in wastage as now there will be 2 ways to create 3.

Teacher: Lets go further

Student: Lets list out all the numbers we can create with just these 3 objects

1 = fork

2 = spoon

3 = fork + spoon

4 = knife

5 = knife + fork

6 = knife + spoon

7 = knife + spoon + fork

We also know that 7 is an upper bound. We know this by adding all the values together. So, 7 is also the least upper bound, and 7 is the furthest we can get. Also, there is no wastage since there are no two combinations which result in the same number

Teacher: I'm guessing then that the tennis ball will now have value 8?

Student: Yes! Now, we can get to 15

Teacher: Ok… whats next?

Student: The marble is now worth 16. Give us some time to figure out how far that gets us.

Teacher: Take your time!

S (after a few minutes): It takes us to 31 without any wastage

Teacher: Okay, is there any way, by changing the values, to go any further?

Student: Don't think so!

Teacher: What makes you say that.

Student: If there is no wastage, it makes sense that we have the best possible solution.

Teacher: You need to form that thought into a much clearer argument. Take your time on that and lets leave this here for now

Task 2

Teacher: Playing the same game as in Task 1, can you come up with a general solution for any number of objects. Mathematicians try and generalise specific solutions they come up with.

Student: What do you mean by generalise?

Teacher: The answer you got to the previous problem was very specific to one each of five distinct objects. A generalised version of this question would look like: Given n distinct objects, give each of them numerical values such that you can make the most number of consecutive numbers starting with 1 by adding them together.

Student: What makes this generalised?

Teacher: Well, the solution to this problem will also give you a solution to our original problem, right?

Student: Yes, by replacing n with 5

Teacher: However, it will also give you the solution to many other specific problems, such as a problem with six distinct objects or maybe even 432435 distinct objects

Student: I see what you mean now!

Teacher: Can you come up with a solution to this generalised problem?

Student: How do we go about doing that. This seems quite hard.

Teacher: Try and spot a pattern in the specific example you solved in Task 1. If you don't see one, try solving the problem for six distinct objects or seven or eight.

S (After trying out six and seven objects): I now see a pattern. If you add in another object, the value you have to give it is twice the value of the largest valued object before.

Teacher: So, what would a solution to n objects look like?

Student: I don't know how we would show it

Teacher: Does this make sense to you?:

$O_1 = 1$

$O_2 = 2$

$O_3 = 2 \times 2$

$O_4 = 2 \times 2 \times 2$

...

$O_{(n-1)} = 2 \times 2 \times 2 \times ... \times 2$ (with (n-2) 2s) $= 2^{(n-2)}$

$O_n = 2 \times 2 \times 2 \times ... \times 2$ (with (n-1) 2s) $= 2^{(n-1)}$

Student: Kind of. But it doesn't tell you what happens in between 4 and n-1

Teacher: Okay, what about this

$O_i = 2^{(i-1)}$ where i runs from 1 to n

Student: Hm... that makes sense. I can find the value of the 10th object by replacing i with 10, and replace i with 32423 to get the 32423rd object.

Teacher: What you have now is a conjectured solution to the question for n objects. Your job is to make it into a theorem by giving an argument for why it is the best solution.

Student: How do we do that?

Teacher: Think about the concept of wastage once again. We will leave this here for now, and come back to it later.

Task 3

Teacher: Now, I will give you 2 each of the 5 original distinct objects. Objects of the same type have to have the same value. Playing the same game, give the objects values to create the most number of consecutive numbers starting at 1.

Student: If objects of the same type have to have the same value, then there will be some wastage here which we cannot help

Teacher: Yes, wastage is built in to the question. Here, your goal is to minimize the wastage, or even remove all wastage apart from the wastage enforced by the question.

Student: Okay, so we still need one of the objects to have value 1. Let the fork have value 1. That means we now have 2 objects with value 1, so we can create not just 1 but 2 as well.

Teacher: Nice! Go on.

Student: We can't make 3, so let the spoons have value 3.

Teacher: Now, how far can you get? Use the upper bound notion from earlier to cap it

Student: We can get to 8, so let the next one be 9... Wait a minute! 9 = 3 x 3. I think this time rather than powers of 2, the solution is powers of 3.

Teacher: Try and see how far powers of 3 gets you.

Student: Here are the values of the 5 object types:

fork = 1

spoon = 3

knife = 9

tennis ball = 27

marble = 81

(After a lot of work) We can reach 242, which is also the least upper bound, and we conjecture that with 2 of each of 5 objects, we can go no further.

Teacher: Interesting conjecture. A proof here is even harder, but once again the concept of wastage is a useful way to think about it.

Task 4

Generalise the solution of Task 3 to 2 each of n distinct objects.

Task 5

Generalise the generalisation in Task 4 to m number of each of n distinct objects.

Task 6

Now, lets return to the situation where you have just 5 types of objects, with 2 of each type. Once again you have to give numerical values to the objects. However, this time, the two objects of the same type do not have the same value, but are additive inverses of each other. So, if one fork has value 5, the other has value -5. With these rules, answer the question.

Task 7

Our goal in all the tasks above was to come up with the best possible solution to a given problem, given some rules. Best possible here means that there exists no configuration of objects and corresponding values which results in a larger number of consecutive numbers starting with 1. In each of the cases above, can you come up with a proof that the solution you proposed is the best possible solution?

Note: Once again, the concept of wastage can be of some use. If you can show your solution has no wastage, that solution must be the best possible solution. This is true because each of the examples above have finite numbers of objects and there are only a finite number of ways to combine them together. If every way of combining them together results in a unique numerical value (no wastage), which is also part of our consecutive number sequence starting with 1, that solution must be the best possible.

Task 8

Now, you have only five of one type of object. However, you also have a set of five differently coloured boxes (Lets say a red box, a green box, a yellow box, a blue box and a black box) which are attached to the floor in a row (red followed by green followed by yellow followed by blue followed by black). You can't lift up the boxes. Using the boxes and the objects, can you come up with a way of creating the most number of consecutive numbers beginning with 1.

Task 9

Now, you have five each of two types of object. However, you also have a set of five differently coloured boxes (Lets say a red box, a green box, a yellow box, a blue box and a black box) which are attached to the floor in a row (red followed by green followed by yellow followed by blue followed by black). You can't lift up the boxes. Using the boxes and the objects, can you come up with a way of creating the most number of consecutive numbers beginning with 1.

Task 10

Now, you have five each of n types of object. However, you also have a set of five differently coloured boxes (Lets say a red box, a green box, a yellow box, a blue box and a black box) which are attached to the floor in a row (red followed by green followed by yellow followed by blue followed by black). You can't lift up the boxes. Using the boxes and the objects, can you come up with a way of creating the most number of consecutive numbers beginning with 1.

Task 11

Now, you have m of only one type of object. However, you also have a set of m differently coloured boxes which are attached to the floor in a row. You can't lift up the boxes. Using the boxes and the objects, can you come up with a way of creating the most number of consecutive numbers beginning with 1.

Task 12

Now, you have m each of 2 types of object. However, you also have a set of m differently coloured boxes which are attached to the floor in a row. You can't lift up the boxes. Using the boxes and the objects, can you come up with a way of creating the most number of consecutive numbers beginning with 1.

Task 13

Now, you have m each of n types of object. However, you also have a set of m differently coloured boxes which are attached to the floor in a row. You can't lift up the boxes. Using the boxes and the objects, can you come up with a way of creating the most number of consecutive numbers beginning with 1.

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