A Conceptual Introduction to Representing Positive Integers

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You will probably agree that 6, 32 and 9873 are numbers while H, J, V, C, R, X and S are letters. However, is that true? You might have heard of Roman Numerals. For the Ancient Romans, V, C and X were numbers, equivalent to 5, 100 and 10 respectively in our current system.

A child might draw what looks like 2 when asked to draw a swan. Is the child saying that the swan is a number? Similarly, a drawing of an egg can look like a 0. Is the drawing of an egg actually the drawing of a number? Clearly not. So, numbers are not the markings on the page. 6 is not a number and neither is V. Both ***can be*** representations of numbers.

Representation and Abstraction

When you draw the face of a person on a piece of paper, there is a lot you can tell about the person by looking at the picture. You might be able to identify the person in a crowd. You might be able to tell their gender, their skin-tone, or whether they have nose hair. However, if you punch the picture, the picture doesn't feel pain. Neither does the person who was represented by the picture (at least not physical pain!). So, the picture is not the person, but a representation of the person.

A picture is also an ***abstraction*** of the person. Abstraction here means that the picture leaves out certain details. It is also a two dimensional representation of a three dimensional object so is bound to leave out certain details. The picture of a face may not tell you how wide the person's face is. However, if someone points to a picture of a frog and of a person and asks you, 'which one of those is a person?,' your reply will be that the latter is. What you are saying here is not that the picture is a person but that the picture is a picture of a person. This is similar to what happened above when we said 6, 32 and 9873 are numbers but H, J, C, R and X are not. The former represent numbers in a particular number system while the latter do not represent numbers in that number system.

Now, don't think of a picture of a particular person, but a child's black and white stick figure which is meant to represent any person. Here, the child has left out a lot of details such as skin tone, shape of the ears, or whether the person has nose hair. So, what the child is doing here is abstracting certain ***traits*** of humans which the child finds common among humans they have seen. The stick figure is an abstract representation of any human.

You can think of numbers in a similar way. What do two horses and two tables have in common? What do seven frogs, seven planets, seven planetary systems and seven buildings have in common? What is common is the concept of two and seven. We have abstracted away from planets, frogs and tables. This allows us to answer questions such as 'are there more frogs in my house or more horses?'

It is important at this point to see that you can never actually see a number or touch it or feel it. You might say that you can see three cows or ten oranges. However, if I ask you to show me three or ten, you will not be able to do so. Showing me the numeral 3 or 10, as we have established, is not good enough since they are representations of numbers rather than numbers.

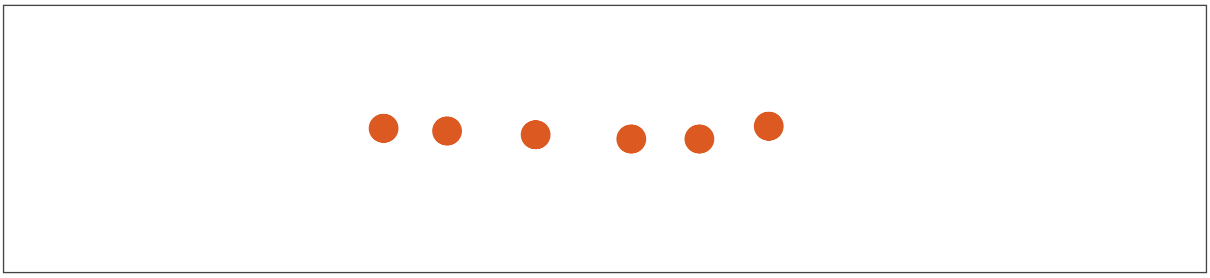
Various Representations of Numbers

There are many ways to represent a particular number. When you were younger, you might have been taught numbers through dots or lines, each representing the number one. Later, you were probably taught the Hindu Arabic number system. This is the number system where 6 represents six and 894 represents the number eight hundred and ninety four. You might have even been taught how to use the Roman Numerals mentioned above. You might also have learnt number names in various languages. Six, seis, ستة, छह, 六, זעקס, and έξι all represent the same number in English, Spanish, Arabic, Hindi, Mandarin, Yiddish and Greek respectively. There is no 'correct' representation. You use one representation rather than another for various reasons including social acceptance and ease of use.

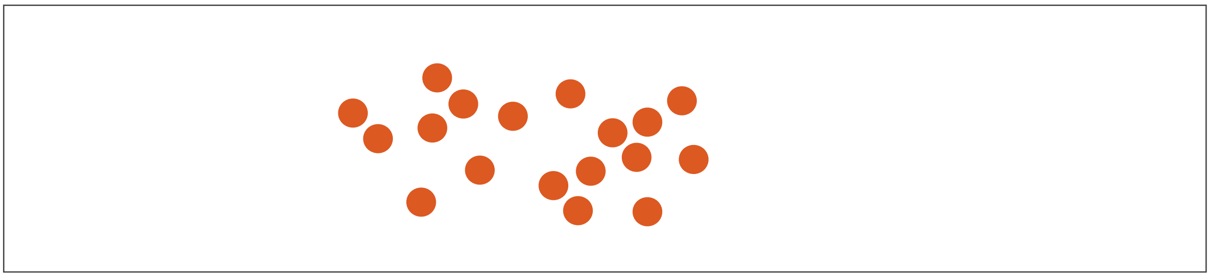
Returning to Dots

Let us look again at the dots you used in primary school. Every number can be represented using these dots (we are only concerning ourselves with positive whole numbers here). Dots/lines are the most obvious representations for numbers. The earliest found markings which archaeologists consider to be numbers are scratched lines. So, why did we create newer representations of numbers? Were dots and lines not good enough?

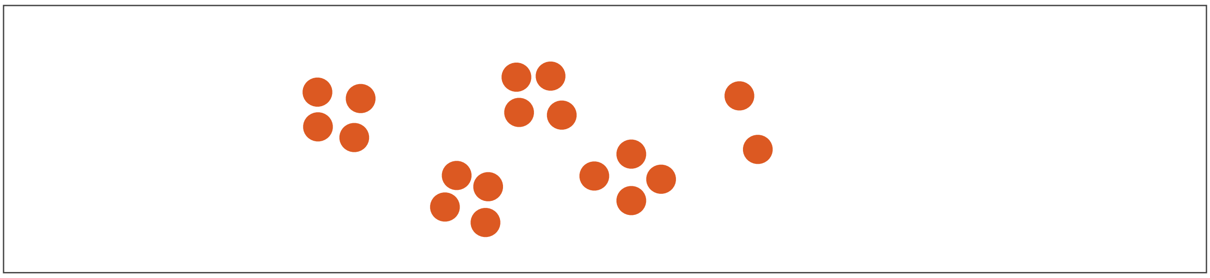
Look below. How many dots do you see (don't count)?



Now, how many dots do you see (don't count)?

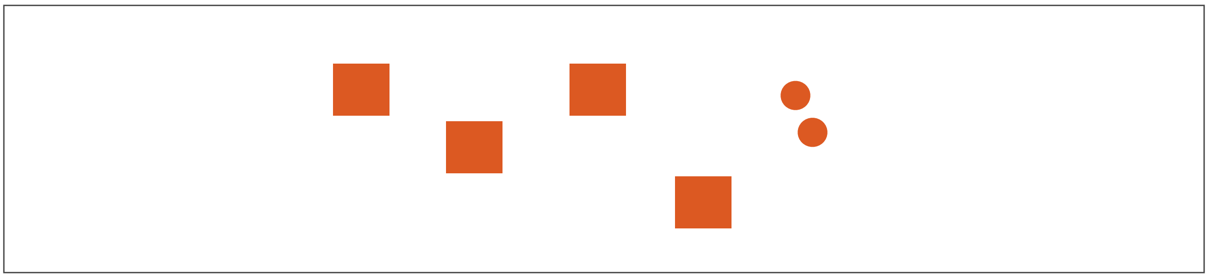


Most adults are able to answer correctly in the first case while almost none can in the second. There are very young children who can answer correctly in the second case. However, even the best of them fail after about a hundred dots (The practice of being able to accurately tell how many objects there are by just looking is called Subitization. Look it up on Duck Duck Go (or Google if you don't mind the NSA watching you) to learn more). Counting more than thirty dots is a chore most people would rather avoid. Given the large numbers we see around us, using dots to represent numbers is out of the question. So, what humans did when numbers got out of hand was to bunch things together and represent a bunch of ones with a different symbol.



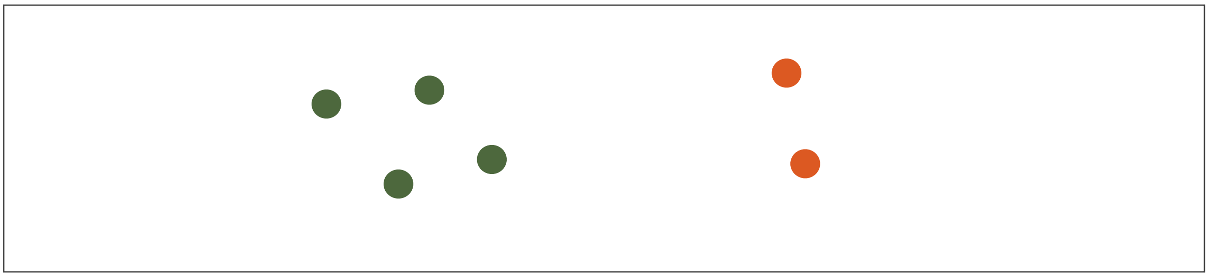
Atomic and Complex Symbols

The picture above has the same number of dots as the previous one. However, you can easily tell how many there are since the dots have been bunched into four units of four dots each, with two left over. Just by bunching them together, we were able to make our lives easier. Here each dot is an 'atomic symbol,' but the collection of four dots is a 'complex symbol.' The value of the complex symbols is derived from the atomic symbols it is made up of. However, I still had to draw each dot (at least copy and paste them!), which took some effort. Now, how about if I do this:



This time, rather than drawing eighteen objects, I only needed to draw six. As you can probably tell, each box represents four dots. What this gives us is two atomic symbols, the square and the dot. The value of the square is not given by its shape or its make-up, like was the case with the bunch of four dots. Rather, it has a value given to it. To take this further, what you can do is to use a triangle to represent, say, twenty dots and so on.

However, as your numbers get bigger and bigger, the number of shapes will get larger and larger and remembering them will become a chore. Another option is to change the colour rather than shape. Let the green dots be equivalent to the square in the picture above:



Even here the same problems of remembering colours will eventually hit you. In every representation, you are playing a game of give and take. In the case of just the dots, there was only one symbol/colour to remember. However, in order to make our work easier, we had to increase the number of symbols/colours and hence our need to memorise. Representations almost always require compromise.

We need to create a more efficient method of representation. When you create a representation, there are many factors you can use to simplify it. The first, which we have used above, is shape. Then is positioning. Notice, that in the first, second and fourth cases above, it doesn't matter what the position of the shapes is. In the third it does, but only to make it easier to count. How do we use the relative position of symbols to simplify the representation? Here, relative means in relation to each other.

Roman numerals are an example of a system which makes use of position to an extent. For all those who have not used Roman Numerals before, here is a sketch of how they work:

The Romans are Coming

Here are what some of the symbols are equal to in Roman-Arabic numerals:

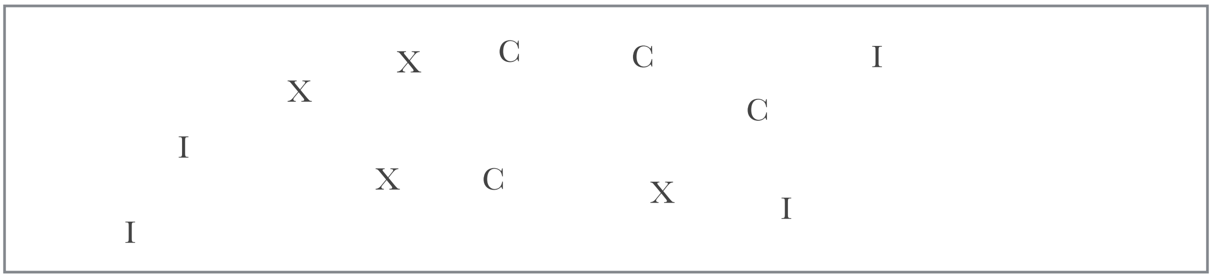
Roman Symbol	Hindu Arabic Translation
I	1
V	5
X	10
L	50
C	100
D	500
M	1000

Here are some numbers represented using the symbols above.

Roman Number	Hindu Arabic Translation
I	1
II	2
III	3
IV	4
V	5
VI	6
VII	7
VIII	8
IX	9
X	10

Initially, the numbers appear to work like our dots. The I is equivalent to the dot. It is an 'atomic symbol.' In fact all of the symbols in the first table are 'atomic symbols.' 2 and 3 and 4 are represented by complex symbols, atomic symbols put together, and so are 6, 7, 8 and 9.

Clearly, roman numerals are using a similar approach to the one we used when we used squares as fours. If they used them in the same way, here would be how they could represent 444.



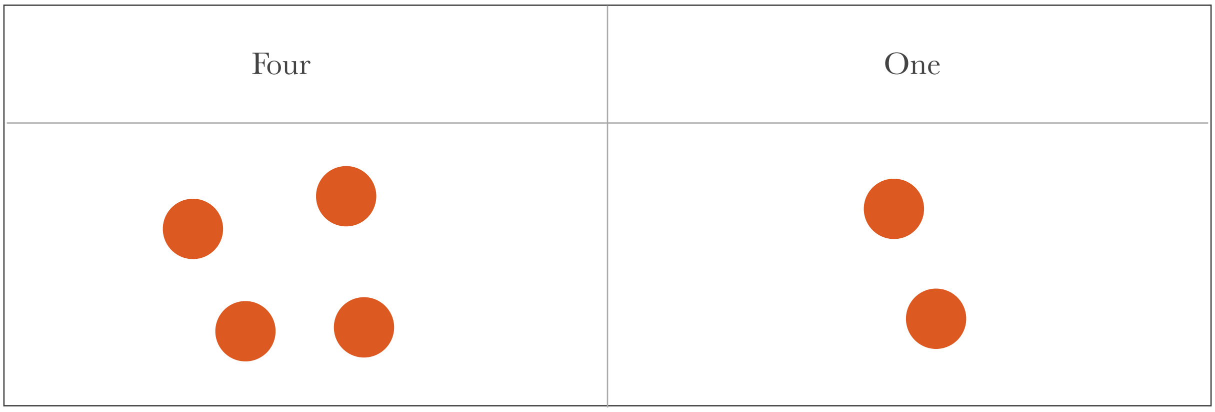
This is not using position at all. However, Roman Numerals do not work exactly this way. In Roman Numerals, XI is 11 but IX is 9. So, putting a number to the left of another subtracts it away. However, this is not always the case. IIX is not 8 and IL is not 49. 8 is VIII while 49 is XLIX, and IIX and IL do not represent any number. To understand exactly how Roman Numerals work, you should look them up. The point of this was to make you see that there is a lot of redundancy in Roman Numerals.

Roman Numeral might make some use of relative position. However, we saw that IIX and IL do not represent any numbers even though they are readable and concise representations. So, there must be a ways of utilising these and similar representations to make Roman Numerals more efficient. One way is to allow both IIX and VIII to represent eight. However, having more than one representation for a particular number could cause confusion. Also, it once again would demonstrate waste since one of those representations could have represented a different number.

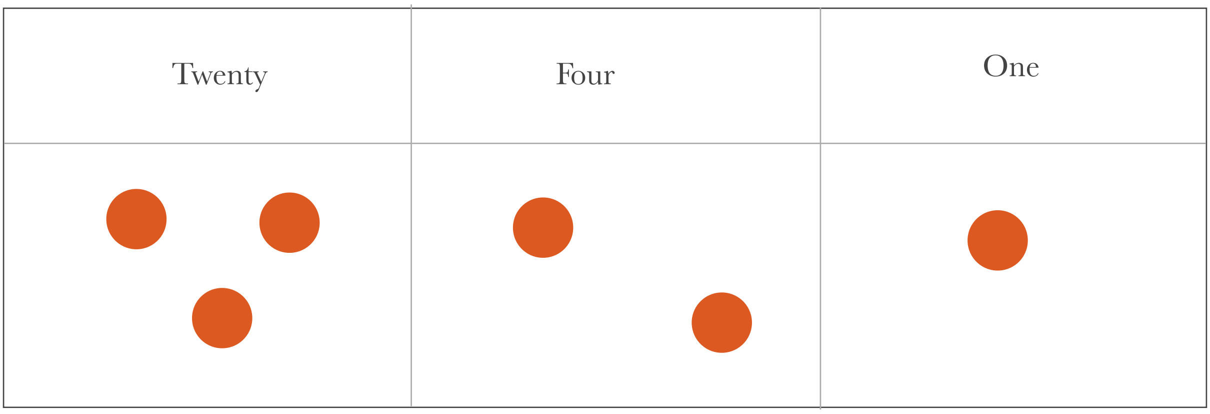
Apart from this deficiency with Roman Numerals, there are others. So far, we have only looked at identifying numbers. Suppose now, we had to perform operations on these numbers - add, subtract, multiply and divide them. Adding is still possible and subtracting is doable, but multiplying and dividing are close to impossible! Try reading up on Roman Numerals and try to do these operations on them to convince yourself of this.

Place Value System

The Place Value system we use today was a conceptual breakthrough. Rather than having different symbols for different values, the place value system uses different positions to represent different values. We can create a place value system using dots to represent the 18 dots we had above:



Similarly 69 can be represented in the following way:

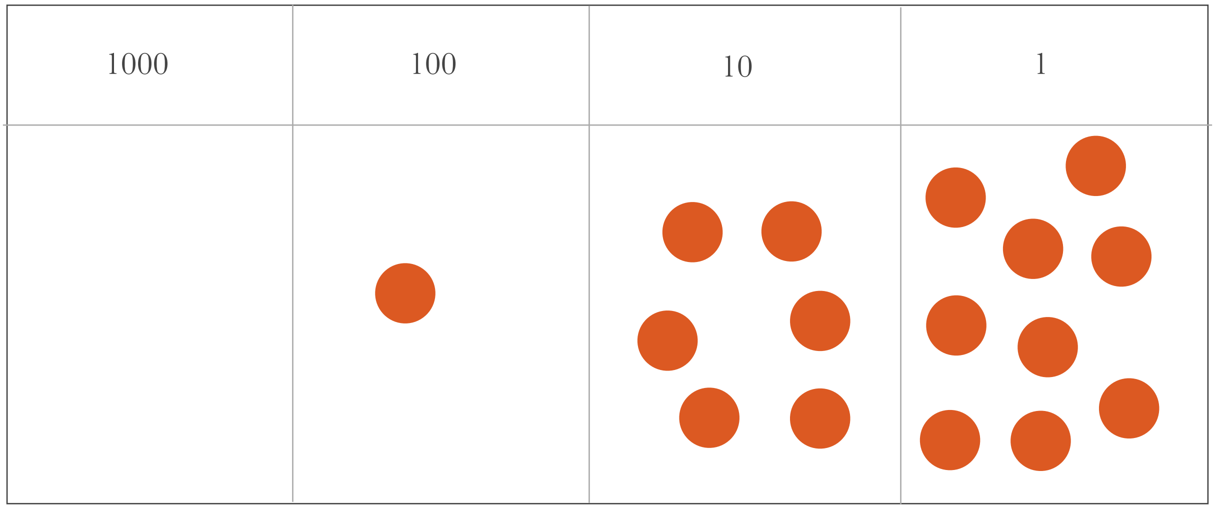


Here, as you can see, all we need is one symbol. However, we still need to write what each of the places represents on the top of the table (or remember it). So, can we create a system where the headers of the table are obvious or are easy to figure out?

The obvious way to do that is to choose values such that there is a relationship between the position and the value - this is why it is called a place-value system. The problem with our choices above is that they appear arbitrary. Why did we pick a value of 20 and not of 24, 18, 16 or 19? Since it is arbitrary, we don't have a way to decide what the next positional value should be.

Base 10

The Roman Numerals we saw above had symbols for every power of 10 as well as for five times every power of 10. The Roman system is not a place value system so it doesn't make sense to say what the particular place represents. Sometimes the V is in the right most place while other times, as in the case of VII and VIII, it is not. In the system we use, each position represents a power of 10. From right to left, the positions represent 1, 10, 100, 1000 and so on.



In this case, no matter how large the number, we know how to add columns to the left in order to represent that number. This brings us onto another issue. The number represented above is equivalent to 169. Why did we not put 16 dots in the 10s column rather than 6 dots in the 10s column and one dot in the 100s? The answer to that is simple: We want to minimise the number of dots we use. In fact, the above representation is the most efficient in this system of representation. The next question which arises is: do we have a procedure to come up with the best representation? Suppose rather than being given the number 169 to translate into dots, you were given a bunch of dots. You don't know how many there. How would you put them into this representation system in the most efficient manner?

Think about this procedure:

1. Pick up a dot and put it in the right-most column
2. Check how many dots there are in the column in which you last put a dot: If there are less than ten dots, return to step 1. If there are ten dots, remove all the dots from that column and add a dot to the column to its left, and repeat step 2.

Try out this algorithm and see if it works. We call this system of representation Base 10 since every place represents a power of 10. The rightmost place is 10^0, then 10^1, 10^2 and so on.

Base 10 with Numerals

We don't tend to use dots to represent numbers, both because it would take a lot of time and because of the space it would take. Rather, we use weird shapes like 4, 7 and 3. However, now we have ten shapes to remember in order to represent positive integers rather than the one shape we had above. So, we have chosen to compromise on the number of shapes we have to remember in order to save on space and energy.

Why do we represent numbers in base 10? There is no real reason apart from convention. Various cultures have used various different bases for different purposes. We currently use a base 7 system for weeks and certain existing measurement systems such as feet, are base 12 (12 inches make 1 foot). Even our conventional measurement of time is not base 10, rather it is base 60 while our measurement of angle is out of 360. Computers use base 2. The babylonians used to use base 12 or 60 while the Mayans used an interesting mixture of base 20 and base 5.

Conclusion

It is important to realise the various ways of representing positive integers to ensure we do not get wedded to a particular representation, and perceive numbers in a particular way. It doesn't really matter from the point of view of mathematics what representation we use since the underlying concept we use is equivalent, but it is important to be able to switch between representations and create new representations. To highlight why, if we were still using Roman Numerals, basic arithmetic would be a chore.

For comments/criticism on this document, please email Madhav Kaushish (madhav@schoolofthinq.com)