Straight Lines & Intersections

Madhav Kaushish

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This is a series of lesson plans on creating a mathematical world, coming up with conjectures in that world, and proving them or finding counter examples. The following are the learning outcomes:

1. An understanding of axioms, conjectures, theorems, and proofs
2. Stating axioms
3. Defining objects
4. Coming up with conjectures
5. Generalizing questions, conjectures, and theorems
6. Coming up with proofs
7. Extending theorems
8. Moving between mathematical worlds

This is broken into two parts, both of which should be done in class. The rest of the problem can be set as a project for students to work on. It is unlikely that a high school student will be able to finish it on their own.

Part 1

Teacher: We will be working on a flat surface. What you have in this world are points and straight lines. We have two possibilities of worlds based on their size:

1. a finite world with definite boundaries, like a sheet of paper.

2. an infinite flat world which goes on forever in all directions

We also have three possibilities, based on the length of the straight lines

a)only finite straight lines allowed (what are also called line segments)

b) only infinite straight lines allowed, which go on forever in both directions

c) both finite and infinite lines allowed

If not specified later, by default, assume we are working in a world which satisfies condition 2 and condition C. These are some of the axioms of this world. An axiom is a statement we assume to be true and work from there.

I am setting one other condition in this world: not more than two lines can intersect at a given point. This gives us another axiom. Assuming the conditions set so far, let me ask you a questions: Given 4575 straight lines and exactly 25 points of intersection per line, what are the total number of points of intersection?

Student: This seems like a very hard problem. It will take too long to figure it out.

Teacher: What a mathematician would do when confronted with such a problem, would be to first generalize it. What that means is that rather than solving a problem about 4575 lines and 25 points of intersection per line, we try to solve the problem about n lines and i points of intersection per line.

Student: That looks nicer, but seems even more difficult. At least with the large number of lines, we could spend a long time and eventually come up with solution.

Teacher: Well, what mathematicians are looking for are general patterns, such that our specific problem becomes much easier to solve. At this stage, a mathematician would try out simple examples of n lines and i points of intersection per line. All we can do is to hope that there is some general pattern which will help us with the specific example at hand.

Student: So, you mean trying out values of n and i like 2 and 1 and so on?

Teacher: Yes. Start by trying out 1 line with 0 points of intersection per line and one line with no points of intersection per line.

Student: The first one has zero points of intersection.

Teacher: How do you know that?

Student: Well there is only one line, and that line has zero points of intersection, so the total number of points of intersection must be zero, right?

Teacher: Sure, so lets move on to the second example.

Student: Hmm.. I don't think the second example is possible to make.

Teacher: Why is that?

Student: Since there is only one straight line, and straight lines cannot intersect with themselves, one straight line can have at most zero points of intersection.

Teacher: How do you know that straight lines cannot intersect with themselves?

Student: Isn't that obvious?

Teacher: In mathematics, the word obvious is not allowed. You have to give reasons for everything.

Student: I can't think of any reason, so can't we just let this be one of those axiom things?

Teacher: Yes we can, for now. However, spend some time thinking about how you can define straight lines. Lets leave that for a different lesson.

Student: So, it seems like there might be some examples where it is not even possible to create the configuration - forget about counting the number of points of intersection. Maybe the example you gave initially is of the same sort?

Teacher: Maybe. So, it might be time to change the question we are asking. As we have seen, we should first be asking a related but similar question: For what values of n and i is it possible to create a configuration of n lines with exactly i points of intersections per line? How about n=1 and i=2?

Student: That is not possible. We already showed it earlier when we said that one line can have most zero points of intersection per line.

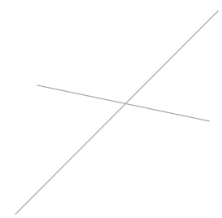
Teacher: Great! What you are quoting from earlier is a theorem. Lets state it clearly:

Theorem 1: Given any configuration with exactly one straight line, there are zero points of intersection in this configuration.

Student: So, can we state this new result also as a theorem?

Teacher: Yes. A theorem is any statement you have proved from the definitions and axioms. Theorem 2: If n=1 and i is greater than or equal to 1, then a configuration with n lines and i points of intersection per line is impossible to create.

Student: So, lets move on to more examples. How about n=2 and i =1. That one clearly is possible. Just take two straight lines and make them intersect.



Teacher: Great!

Theorem 3: A configuration with 2 lines and 1 point of intersection per line is possible to create.

Student: However, 2 lines and 2 points of intersection per line seems to be impossible.

Teacher: For now, that counts as a conjecture. A conjecture is a statement without a proof. When you prove a conjecture, you get a theorem.

Student: Well, start with a line. That line cannot intersect with itself from, which we get from Theorem 1. So, it has to intersect with two other lines. There is only one more line allowed, so this configuration is impossible.

Teacher: Why does it have to intersect with two other lines? Why can't it intersect with one other line twice?

Student: Thats impossible. But, I can't prove it. So, lets make it another axiom?

Teacher: Sure!

Theorem 4: If n=2 and i is equal to 2, then a configuration with n lines and i points of intersection per line is impossible to create.

Student: I'm beginning to have a suspicion here that n has to be greater than i.

Teacher: Another conjecture! Can you prove it?

Student: I think so. From our axioms, we already know that one line cannot intersect with itself. Also, two lines can have at most one point of intersection. Given n lines, pick a line. There are only n-1 lines left. So, the line you picked can have at most n-1 points of intersections.

Teacher: Fantastic! Do you realize, you have done what seems to be infinite work in a finite amount of time! You have not only proved something for n=2 and i=2, but also for n=321 and i =342423, and an infinite number of other cases. One of the greatest parts of being a mathematician is that your goal is to do the most amount of work with the least amount of effort.

Theorem 5: If $i \geq n$, then a configuration with n lines and i points of intersection per line is impossible to create.

Student: So, lets explore some more examples.

Teacher: Your homework is to try out the following:

S. No.	*n*	*i*	*Possible?*	*S. No.*	*n*	*i*	*Possible?*
3	3	0		**13**	5	3	
4	3	1		**14**	5	4	
5	3	2		**15**	6	0	
6	4	0		**16**	6	1	
7	4	1		**17**	6	2	
8	4	2		**18**	6	3	
9	4	3		**19**	6	4	
10	5	0		**20**	6	5	
11	5	1					
12	5	2					

Part 2

Teacher: Did you figure the homework out?

Student: A few of them. I think I can state a conjecture: A configuration with n lines and zero points of intersection per line is always possible to create.

Teacher: How would you go about constructing such a configuration?

Student: Just take n finite straight lines which do not intersect. You can do it with infinite lines as well, but this is easier.

Teacher: Great.. Another theorem

Theorem 6: A configuration with n lines and zero points of intersection per line is always possible to create.

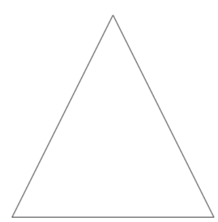
Student: I don't think I have any more general ones.

Teacher: Thats fine. Lets go through the list one by one. Maybe, we will see a pattern. Before we do that, lets introduce some notation. Lets say (n,i) represents a configuration with n lines and i points of intersection per line. So, lets start with (3,1).

Student: I haven't been able to create it. I think its impossible, but don't have a proof.

Teacher: Lets come back to that later. How about (3,2).

Student: Thats possible. Just draw a triangle.

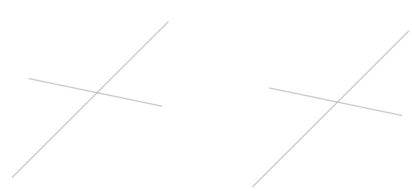


Teacher: Theorem 7: (3,2) is possible.

Student: (4,1) is also possible

Teacher: How?

Student: By drawing two crosses:



Teacher: Theorem 8: (4,1) is possible to create If we restricted ourselves to only infinite lines, would it still be possible?

Student: I don't think so

Teacher: I'm going to leave that to you to figure out why

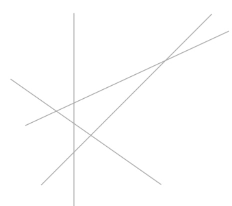
Student: (4,2) is also possible by making a square, and I think I've found a pattern. (n,2) is always possible for all n¿2

Teacher: How would you make (n,2) for an arbitrary n

Student: By making an n-sided polygon

Teacher: Theorem 9: (n,2) is possible to create for all n ¿ 2

Student: (4,3) is also possible:



Teacher: Theorem 10: (4,3) is possible to create

Student: That gives me an idea - I think (n,n-1) is always possible for all n

Teacher: Can you prove that?

Student: Can't you just take n infinite lines? Each one will intersect with all the others resulting in n-1 points of intersection.

Teacher: Can you pick any n lines? Do two infinite lines always intersect with each other?

Student: Oh! Not if they are parallel. So, take n non-parallel lines.

Teacher: For that you need to define parallel lines.

Student: They are lines which do not intersect.

Teacher: Lets see what you are saying:

I asked whether infinite lines always intersect. You replied that they do not do so if they are parallel. Then when I asked you what parallel lines are you replied that they are lines which do not intersect. The reasoning here seems circular, doesn't it?

Student: Then, can't we say parallel lines are those lines which are equidistant.

Teacher: In that case, what is distance? We don't have a way to measure that in this world

Student: Then how do we deal with this?

Teacher: Mathematicians have developed sophisticated ways to introduce the concept of distance to such worlds. However, we can play a trick Euclid, the first real mathematician we know of, often played. We can say parallel lines are undefined entities (what are also called primitives) with the following properties (which are also axioms of the world):

given any straight line, there are infinite parallel lines to it

parallel lines do not intersect with each other

infinite non-parallel lines always intersect with each other

Student: We will also need to add in another axiom: given a straight line, there are infinite non-parallel lines to it. That should give us a proof that (n,n-1) is possible for all n.

Teacher: There is still one problem. Remember that our world requires at most two straight lines to intersect at a given point. We need to show that it is possible to avoid this when creating the (n,n-1) configuration. Lets leave that aside for now and let (n,n-1) be a theorem contingent on us proving this intermediate result:

Theorem 11: (n,n-1) is possible to create for all $n > 0$

Student: So far, apart from when $i \geq n$, we have proved everything else so far as possible. Maybe (n,i) is always possible if $i < n$.

Teacher: Thats another conjecture. However, is it true? Earlier, you said (3,1) might not be possible to create. Can you create it?

Student: No. Not so far.. Actually, I don't think its possible

Teacher: Can you prove that?

Student: Well, take two lines and make them intersect. You have one left over. That line has nothing to intersect with. So, (3,1) is not possible.

Teacher: I don't think that reasoning works. What you need to prove is that no matter what you do, (3,1) is impossible to create. You have given one method and shown it is impossible using that method. Maybe, there is some other way to accomplish it.

Student: I don't think there is any way. How does this reasoning sound:

Take a line. We need it to have one point of intersection. Since it cannot intersect with itself, it will have to intersect with one of the other two lines. Make that happen. Now, two lines have exactly one point of intersection. We know that since we have an axiom that two lines can intersect at most at one point. We are left with one line. That line cannot intersect with any of the other two lines since if it did, at least on line would have two points of intersection. So, there is no way for that line to have more than zero points of intersection. Hence, (3,1) is not possible to create.

Teacher: Great! As you might have noticed, in mathematics, proving something to not be possible is usually much harder than proving that it is possible.

Student: I think I have found another pattern - (n,1) is possible if n is even and (n,1) is impossible if n is odd.

Teacher: Let me introduce some terminology. When mathematicians state something like you have, they usually use the phrase 'if and only if.' So, your claim would become:

(n,1) is possible to create if and only if n is even. What this means is two things:

(n,1) is possible to create if n is even

If (n,1) is possible to create, then n is not even

This is the same as what you said. Can you come up with a proof?

Student: To show that (n,1) is possible if n is even, you can just create n/2 crosses with finite lines which do not intersect with each other. To show (n,1) is impossible if n is odd, we will have to show that you are forced to create crosses till you have just one line left.

Teacher: Let me introduce you to a systematic way of doing exactly that. It is called proof by induction:

I'm not sure whether you have encountered dominos. Not the pizza place, but the little tokens. The goal there is to place them in such a way that when you knock the first one over, all the rest get knocked over in a sort of chain reaction.

In proof by induction, you establish a base case (the equivalent of the first domino), and then show that if the k-th case is true, than the (k+1)-th case is also true. This will show that if the first case is true then the second is true. If the second is true then the third is true, and so on.

In this example, your base case is that (3,1) is impossible to create. You have already proved that.

Now, we assume (k,1) is impossible where k is odd. The next odd number after k is k+2. Lets try creating (k+2,1). Take a line. Using the same reasoning as the (3,1) case, that line has to intersect with another line. So, we are left with k lines, none of which can intersect with the two lines we picked first. So, we have to create (k,1). However, we have assumed (k,1) to be impossible. Hence we have established that if (k,1) is impossible then (k+2,1) is impossible. Put together with the base case, this gives us a proof.

This is a hard method of proof, and you will have to see various examples of it before you actually internalize what is going on.